1. Let \( g \) be the ABACAB function of three \( n \times n \) matrices:

\[
g(A, B, C) = ABACAB
\]

a [5 pts]. If you wrote down the Jacobian of \( g \) – but don’t! – how big a matrix would it be (what by what)?

**Answer.** \( 3n^2 \times n^2 \).

[g eats three \( n \times n \) matrices and produces one.]

1b [15 pts]. Compute the derivative of \( g \), at every \( (A, B, C) \). Your answer should (had better) be a linear transformation of its inputs.

**Answer.** Let’s perturb by \( (H_A, H_B, H_C) \). Then

\[
\lim_{\varepsilon \to 0} \frac{g(A + \varepsilon H_A, B + \varepsilon H_B, C + \varepsilon H_C) - g(A, B, C)}{\varepsilon}
\]

\[
= \lim_{\varepsilon \to 0} \frac{ABACAB + \varepsilon (H_A BACAB + A H_B ACAB + A B H_A CAB + A B A H_C AB + A B A C A H_B B + A B A C A H_B)}{\varepsilon}
\]

\[
= H_A BACAB + A H_B ACAB + A B H_A CAB + A B A H_C AB + A B A C A H_B + A B A C A H_B
\]

i.e. the derivative at \( (A, B, C) \) is the linear transformation

\( (H_A, H_B, H_C) \mapsto H_A BACAB + A H_B ACAB + A B H_A CAB + A B A H_C AB + A B A C A H_B + A B A C A H_B \)

[Sanity check: consider \( n = 1 \) where things commute, so \( g(A, B, C) = A^3 B^2 C \), and this would give \( 3A^2 B^2 C H_A + 2A^3 B C H_B + A^3 B^2 C \) Then the partial derivatives are \( \frac{da}{dA} = 3A^2 B^2 C, \frac{da}{dB} = 2A^3 B C, \frac{da}{dC} = A^3 B^2 \), which are indeed the coefficients in front of \( H_A, H_B, H_C \).]
2. Let \( C \subseteq \mathbb{R}^n \) be an open set containing \( \vec{0} \), and for \( \lambda \in \mathbb{R}_+ \) and \( \vec{u} \in \mathbb{R}^n \), define
\[
\vec{u} + \lambda C := \{ \vec{u} + \lambda \vec{c} : \vec{c} \in C \}.
\]
Call a set \( U \subseteq \mathbb{R}^n \) \( C\)-open if for every \( \vec{u} \in U \), there exists a \( \lambda > 0 \) such that \( U \supseteq \vec{u} + \lambda C \). We're given that \( \vec{u} + \lambda C \subseteq U \), so for any \( \vec{u} + \lambda C \subseteq U \), there exists a \( \lambda > 0 \) such that \( \vec{u} + \lambda C \subseteq U \). And by the way that's \( \vec{u} + rB_1(\vec{0}) \).

**Answer.** We need to show that for any \( \vec{u} \in U \), there exists a ball \( B_{r>0}(\vec{u}) \subseteq U \). And by the way that's \( \vec{u} + rB_1(\vec{0}) \).

What we're given is that there exists a \( \vec{u} + \lambda C \subseteq U \). So if we can find a ball \( \vec{u} + rB_1(\vec{0}) \subseteq \vec{u} + \lambda C \), i.e. \( rB_1(\vec{0}) \subseteq \lambda C \), then we're done.

Since \( C \) is open and \( \vec{0} \in C \), we know there's a ball \( B_r(\vec{0}) \subseteq C \). Scaling the vectors in both, we know \( \lambda B_r(\vec{0}) \subseteq \lambda C \). So if we can find a ball \( rB_1(\vec{0}) \subseteq \lambda B_r(\vec{0}) \), then we're done.

That's easy: take \( r = \lambda s \). Then those balls are equal.

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2b [20 pts]. If \( C \) is bounded, and \( U \) is open, prove \( U \) is \( C \)-open.

**Answer.** We need to show that for any \( \vec{u} \in U \), there exists a \( \lambda > 0 \) with \( \vec{u} + \lambda C \subseteq U \).

We're given that \( C \) is bounded, i.e., there exists a radius \( R > 0 \) such that \( C \subseteq B_R(\vec{0}) \). Also that \( U \) is open, so there exists a ball \( B_s(\vec{u}) \subseteq U \) for some \( s > 0 \).

So we'd like to choose \( \lambda \) so that \( \vec{u} + \lambda C \subseteq B_s(\vec{u}) \). We know \( \vec{u} + \lambda C \subseteq \vec{u} + \lambda B_{10}(\vec{0}) = B_{\lambda R}(\vec{u}) \), so it's enough to get \( B_{\lambda R}(\vec{u}) \subseteq B_s(\vec{u}) \). As in (2a), that's easy: take \( \lambda = s/R \).

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2c [15 pts]. Give an example of a \( C \) and an open set \( U \) that isn't \( C \)-open.

**Answer.** By (2b), we need \( C \) unbounded. The dumbest example is \( C = \mathbb{R}^n \) (for \( n > 0 \)). Then every \( \vec{u} + \lambda C = \mathbb{R}^n \), too. So as long as \( U \neq \emptyset, \mathbb{R}^n \), it's a counterexample [but you can't stop there! e.g. for \( n \) there is no such \( U \); let's say \( U = (0, 1) \subseteq \mathbb{R}^1 \)].

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3. Let \( A \) be an \( n \times n \) matrix.

For \( S \subseteq \mathbb{R}^n \) a linear subspace, define
\[
S^* := \{ \vec{v} \in \mathbb{R}^n : \forall \vec{s} \in S, \vec{s} \cdot A\vec{v} = 0 \}
\]
a [10 pts]. Prove that \( S^* \) is a linear subspace too. (Meaning: show that it satisfies the short list of requirements.)

**Answer.** There are three requirements: it should have \( \vec{0} \), be closed under multiplication by any scalar, and closed under addition.

\[
\forall \vec{s} \in S, \vec{s} \cdot A\vec{0} = \vec{s} \cdot \vec{0} = 0 \quad \checkmark
\]

If \( \vec{v} \in S^* \) and \( c \in \mathbb{R} \), then \( \forall \vec{s} \in S \), we have
\[
\vec{s} \cdot A(c\vec{v}) = \vec{s} \cdot cA\vec{v} = c \vec{s} \cdot A\vec{v} = c \vec{0} = 0.
\]

So \( c\vec{v} \in S^* \) too. \( \checkmark \)

If \( \vec{v}_1, \vec{v}_2 \in S^* \), then \( \forall \vec{s} \in S \), we have
\[
\vec{s} \cdot A(\vec{v}_1 + \vec{v}_2) = \vec{s} \cdot (A\vec{v}_1 + A\vec{v}_2) = \vec{s} \cdot A\vec{v}_1 + \vec{s} \cdot A\vec{v}_2 = 0 + 0 = 0.
\]

\( \checkmark \)
So $\vec{v}_1 + \vec{v}_2 \in S^\ast$ too. ✓

3b [15 pts]. If $A$ is symmetric, prove that $(S^\ast)^\ast \geq S$. (Meaning: assume $\vec{b} \in S$, and prove $\vec{b} \in (S^\ast)^\ast$.)

**Answer.** Assume $\vec{b} \in S$. We want to know that $\vec{b} \in (S^\ast)^\ast$, i.e. if $\vec{t} \in S^\ast$, then $\vec{t} \cdot A\vec{b} = 0$.

What we know for sure is that $\vec{b} \cdot A\vec{t} = 0$, since $\vec{t} \in S^\ast$.

[Those two equations say different things about $\vec{b}$, since it’s on the right in one and the left in the other. That, and the fact that we’re given $A = A^\top$, suggest using transpose.]

We can rewrite $\vec{b} \cdot A\vec{v}$ as the entry of the $1 \times 1$ matrix $\vec{b}^\top A\vec{v}$, thinking of $\vec{b}, \vec{v}$ as skinny matrices. Then since a $1 \times 1$ matrix is symmetric,

$$\vec{b}^\top A\vec{v} = (\vec{b}^\top A\vec{v})^\top = \vec{v}^\top A^\top \vec{b}$$

then use $A = A^\top$ and learn $\vec{b} \cdot A\vec{v} = \vec{v} \cdot A\vec{b}$.

Since $\vec{b} \cdot A\vec{t} = 0$, and $\vec{b} \cdot A\vec{v} = \vec{v} \cdot A\vec{b}$, we learn $\vec{t} \cdot A\vec{b} = 0$ for all $\vec{t} \in S^\ast$, i.e. $\vec{b} \in (S^\ast)^\ast$.

3c [10 pts]. Give an example of $n$, $A$ symmetric, and $S$ such that $(S^\ast)^\ast \neq S$.

**Answer.** $n = 1$, $A = [0]$, $S = \{\vec{0}\}$. Then $S^\ast = (S^\ast)^\ast = \mathbb{R}^1 > S$.

[Thought process: we know $(S^\ast)^\ast > S$, by (3b). For it to grow in this way, we need being-in-$V^\ast$ to be easy for $V = S, S^\ast$. The $= 0$ condition in the definition of $V^\ast$ gets easier to satisfy as $A\vec{v}$ gets to be $\vec{0}$ more often. The easiest way to ensure that is to take $A$ the zero matrix.

In fact, $(S^\ast)^\ast = S$ for all $S \leq \mathbb{R}^n$ iff $A$ is invertible.]