Numbers in boxes are the point scores for their questions.

1. True or false? If false, give a counterexample. (If you think it’s really obvious that it’s a counterexample, don’t bother proving it.) If true, you don’t have to prove it.

[3 apiece]

Every continuous function is Lebesgue integrable.
A. False; \( f = 1 \) on \( \mathbb{R} \) is not.

Every Riemann integrable function is continuous.
A. False; \( f = 1_{[0,1]} \) is not.

Every Riemann integrable function is Lebesgue integrable.
A. True.

Every Lebesgue integrable function is Riemann integrable.
A. False; \( f = 1_{\mathbb{Q} \cap [0,1]} \) is not.

Every continuous, bounded, Lebesgue integrable function of bounded support is Riemann integrable.
A. True; one doesn’t even need to assume Lebesgue integrable (it follows).

Every Lebesgue integrable function has bounded support.
A. False; \( f = 1_z \) is Lebesgue integrable.

2. [10]

Let \( S \subseteq \mathbb{R}^2 \) be the half-disc defined by \( x^2 + y^2 \leq 1, x \geq 0 \). Integrate the function \( f(x,y) = x \) over it via Fubini’s theorem. Do it in both orders – over \( x \) then \( y \), and vice versa. Hint: the substitution \( z = 1 - x^2 \) will help more than trig substitutions.
A.

\[
\int_{x=0}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x \, dy \, dx = \int_{x=0}^{1} x \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \, dx
\]

\[
= \int_{x=0}^{1} x \cdot 2\sqrt{1-x^2} \, dx
\]

\[
= \int_{x=0}^{1} -\sqrt{z} \, dz
\]

\[
= \int_{z=0}^{1} \sqrt{z} \, dz
\]

\[
= \frac{z^{3/2}}{3/2} \Big|_{0}^{1}
\]

\[
= \frac{2}{3}.
\]

\[
\int_{y=-1}^{1} \int_{x=0}^{\sqrt{1-y^2}} x \, dx \, dy = \int_{y=-1}^{1} \frac{x^2}{2} \sqrt{1-y^2} \, dy
\]

\[
= \int_{y=-1}^{1} \frac{1-y^2}{2} \, dy
\]

\[
= \frac{y-y^3/3}{2} \Big|_{-1}^{1}
\]

\[
= \frac{2}{3}.
\]

3. [10]

Let \( S \subseteq \mathbb{R}^n \) have Hausdorff dimension \( d < n \), i.e. for any \( \epsilon > 0 \), we can cover it fully with cubes of side-lengths \( r_1, r_2, \ldots, \) all \( < \epsilon \), such that \( \sum i r_i^{d+\epsilon} < \epsilon \).

Prove that \( S \) has measure 0.

A. Since \( d < n \), there exists \( \epsilon \in (0, n-d) \), so \( d + \epsilon < n \). If \( r_i < 1 \), then \( r_i^n < r_i^{d+\epsilon} \). Adding them up,

\[
\sum_i r_i^n < \sum_i r_i^{d+\epsilon} < \epsilon
\]

which is the bound we needed to show to say \( S \) has measure 0.
4. Consider, if you will, two Lebesgue integrable functions \( f, g \) that aren’t Riemann integrable, but whose product \( fg \) is.

a. [5] Give an example of such a pair, where \( f, g \) are bounded with bounded support.
A. \( f = 1_{Q \cap [0,1]}, \ g = 1_{Q \cap [2,3]}, \) so \( fg = 0. \)

b. [5] Give an example of such a pair, where \( f, g \) are continuous almost everywhere.
A. \( f = 1_{2\mathbb{Z}}, \ g = 1_{2\mathbb{Z}+1}, \) i.e. on the evens vs. the odds. Again \( fg = 0. \)

c. [5] Can we achieve (b) and (c) with the same pair \( f, g \)?
A. Nope. If \( f \) (or \( g \)) is bounded with bounded support and continuous almost everywhere, it’s Riemann integrable.

5. Let \( M \) be an \( n \times n \) matrix, and \( N \) the matrix obtained by shaving off \( M \)'s last row and column.

a. [5]
If \( M \) is upper triangular, show that \( N \)'s characteristic polynomial \( \det(N - tI_{n-1}) \) divides \( M \)'s.
A. The characteristic polynomial of an upper triangular matrix with diagonal entries \( d_1, \ldots, d_m \) is \( \prod_{i=1}^{m} (d_i - t) \), since the determinant of an upper triangular matrix is the product down the diagonal.
Hence if \( M \)'s diagonal entries are \( m_{11}, \ldots, m_{nn} \), the polynomials in question are \( \prod_{i=1}^{n} (m_{ii} - t) \) and \( \prod_{i=1}^{n-1} (m_{ii} - t) \).

b. [5]
Does this divisibility hold without the upper triangularity assumption?
A. Nope. Perhaps the simplest nontriangular matrix is
\[
M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]
whose characteristic polynomial is \( t^2 - 1 \), which is not a multiple of \( N \)'s, which is just \( t \), or I guess \(-t\), as defined above. (Here \( N = (0) \).)
6. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a smooth function such that

\( (1) \) \( g, h \) are two more smooth functions, with the property that

\[
f = \frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}
\]

\( (2) \) If \( |\vec{v}| > 17 \), then \( |f(\vec{v})| < 17|\vec{v}|^{-17} \), and same for \( g \) and \( h \).

Your goal: show the equality of (Lebesgue) integrals

\[
\int_{x=1}^{\infty} \int_{y=-\infty}^{\infty} f(x, y) \, dx \, dy = \int_{y=-\infty}^{\infty} h(1, y) \, dy.
\]

a. [10] Set this up as a formal (i.e. nonrigorous) application of Stokes’ theorem: say what manifold-with-boundary \( M \) you’re using, what are the forms, etc. Hint: \( g \) and (obviously) \( h \) will come into play.

A. Stokes’ theorem says

\[
\int_M d\alpha = \int_{\partial M} \alpha
\]

so it looks like \( M \) should be the half plane \( \{(x, y) : x \geq 1\} \), and \( d\alpha \) should be \( f \, dx \wedge dy \). What should \( \alpha \) be? Some 1-form, certainly.

A general 1-form is a combination of \( dx \) and \( dy \), say \( \alpha = A \, dx + B \, dy \). Then

\[
d\alpha = dA \wedge dx + dB \wedge dy = \frac{\partial A}{\partial y} \, dy \wedge dx + \frac{\partial B}{\partial x} \, dx \wedge dy = \left( \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) \, dx \wedge dy.
\]

(There are other terms in \( dA \) and \( dB \), but they die upon being wedged with \( dx \) and \( dy \).)

That begins to look like assumption (1), with \( A = f \) and \( B = g \). So now Stokes would say

\[
\int_{(x,y):x \geq 1} f = \int_{(x,y):x = 1} (g \, dx + h \, dy)
\]

but as we traverse the line \( x = 1 \), the tangent vector has no horizontal component, so the \( g \, dx \) term disappears.

Therefore it becomes \( \int_{(x,y):x = 1} h \, dy \), which is what we wanted.
b. [5] Explain why Stokes’ theorem doesn’t *quite* apply as usually stated. Then give whatever additional argument is necessary to complete the proof. (Hint: look at the picture you drew, or should have drawn, in question #2.)

A. $M$ is noncompact. (If we didn’t include compactness in the assumptions of Stokes’ theorem, we could e.g. just rip out all of $\partial M$, which wouldn’t change the $\int_M$ side but would make the $\int_{\partial M}$ side vanish.) So Stokes’ theorem doesn’t actually apply.

Instead, let’s apply it to a large half-disc $M_R$, where $|(x,y)| \leq R$. Then Stokes says

$$\int_{M_R} f \, dx \wedge dy = \int_{x=1, |(x,y)| \leq R} \alpha + \int_{x \geq 1, |(x,y)| = R} \alpha$$

where the RHS has a term for the left edge, and the arc, of the half-disc. Now consider the limit $R \to \infty$. The first two terms become what we want them to. The arc term goes to zero because of the bound (2) on $g$ and $h$. (The length of the arc increases like $R^1$, while $g$ and $h$ decrease like $R^{-17}$.)

7. We know that if $M$ is a compact oriented manifold-with-boundary, then its boundary $\partial M$ is a compact oriented manifold.

So let’s consider the reverse. If $N$ is a compact oriented manifold, is $N = \partial M$ for some $M$? If $N$ is 1-dimensional, it’s a bunch of oriented circles, and **Yes** this $N$ is the boundary of a bunch of discs (or of a sphere with a bunch of holes chopped in it, like a Wiffle® ball).

The same holds if $N$ is 2-d (easy to visualize) or even 3-d (less easy!). The questions below concern $\dim N = 0$.

[4] If you were thinking of a compact oriented 0-manifold $N$, and wanted to tell someone about it over the phone, what data would you tell them to describe it?

A. How many $+$ points and how many $-$ points there are.
[10] State the necessary and sufficient condition for such an \( N \) to be the boundary of a compact 1-manifold \( M \) with boundary.

A. A compact 1-manifold with boundary is a collection of intervals. The boundary of any one is a + point and a - point. So there should be the same number of each.

8. Define \( f : \mathbb{R}^2 \to \mathbb{R}^3 \) by \( f(x, y) = (x, e^x, 1 + y^2) \). Let \( \alpha = u \, du + w \, dv \) be a 1-form on \( \mathbb{R}^3 \), in its \( u, v, w \) coordinate system.

a. [3] Compute the 1-form \( f^*(\alpha) \).

A. \( u \, du + w \, dv \mapsto x \, dx + (1 + y^2) \, d(e^x) = x \, dx + (1 + y^2)e^x \, dx = (x + (1 + y^2)e^x)dx \).

b. [5] Set up and compute its integral over the interval \( x \in [0, 1], y = 1 \).

A.

\[
\int_{x=0}^{1} (x + (1 + y^2)e^x)dx \quad = \quad \int_{x=0}^{1} (x + 2e^x)dx
\]

\[
\quad = \quad \left( \frac{x^2}{2} + 2e^x \right)\bigg|_{0}^{1}
\]

\[
\quad = \quad (1/2 + 2e) - (0 + 2)
\]

\[
\quad = \quad 2e - 3/2.
\]