Exercise 1. Show that certain sets exist. (NS 317 #2)

Solution 1.

(a) By Axiom 2, ∅ exists. Then the Axiom of Unordered Pairs gives that {∅, ∅} exists. Using the Axiom of Extensionality, for any set \( x, \{ x, x \} = \{ x \} \). So, \{ ∅ \} exists.

(b) By Axiom 2 and part (a), ∅ and \{ ∅ \} both exists. Forming the Unordered Pair: \{ ∅, ∅ \} exists.

(c) \{ ∅ \} exists by (a). Take Unordered Pairs and use Extensionality to get \{ \{ ∅ \} \}. Repeat to get \{ \{ \{ ∅ \} \} \}. \{ ∅, ∅ \} exists by (b). Take Unordered Pairs to get \{ \{ ∅, ∅ \}, \{ \{ ∅ \} \} \}. The Union Axiom gives existence of \{ ∅, ∅, \{ ∅ \} \}.

(d) \{ ∅, ∅, \{ ∅ \} \} exists by (c). \{ ∅, ∅ \} exists by (b) and then \{ \{ ∅, ∅ \} \} exists by Unordered Pairs and Extensionality. Unordered Pairs gives \{ \{ ∅, ∅ \}, \{ \{ ∅ \} \} \}. Apply Union to get: \{ ∅, ∅, \{ ∅ \}, \{ ∅, ∅ \} \}.

Exercise 2. Show that certain sets exist. (NS 371 #3)

Solution 2.

(a) As we saw above, if \( x \) exists then applying Unordered Pairs and Extensionality gives the singleton \{ x \}.

(b) Given \( x, y, z \), apply Unordered Pairs to get \{ x, y \}. Apply Unordered Pairs and Extensionality to get \{ z \}. Unordered Pairs gives \{ \{ x, y \}, \{ z \} \}. Union gives \{ x, y, z \}.

(c) We want the set \( w \) such that for any set \( a, a \in w \) iff \( a \in x \) or \( a \in y \). I.e. we want the "union" of all elements in \( x \) and \( y \). So, Unordered Pairs gives \{ x, y \} and then Union gives \( w = \bigcup \{ x, y \} \) as the desired set.

Note: Since the proofs are very similar in style for the following two problems, I am alternating which properties to prove for sets and which for binary relations.

Exercise 3. Verify each of the laws for the Boolean Operations on sets listed as (1A) - (9).

Solution 3.

- Associativity (1A): We want to show that \( (A \cup B) \cup C = A \cup (B \cup C) \). Suppose \( x \in (A \cup B) \cup C \). Then \( x \in (A \cup B) \) or \( x \in C \) (or both). In the first case, \( x \in A \) or \( x \in B \). If \( x \in A \) then \( x \in A \cup (B \cup C) \). Similar reasoning gives that \( x \in A \cup (B \cup C) \). Hence, in either case, \( x \in (A \cup B) \cup C \) implies that \( x \in A \cup (B \cup C) \). We conclude that \( (A \cup B) \cup C \subset (A \cup (B \cup C)) \). Now suppose that \( x \in A \cup (B \cup C) \). Then \( x \in A \) or \( x \in (B \cup C) \). If \( x \in A \) then \( x \in (A \cup B) \) and so \( x \in (A \cup B) \cup C \).

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If \( x \in (B \cup C) \) then \( x \in B \) or \( x \in C \). If \( x \in B \) then \( x \in (A \cup B) \) and \( x \in (A \cup B) \cup C \). Finally, if \( x \in C \) then \( x \in (A \cup B) \cup C \). Hence, in all cases, \( x \in A \cup (B \cup C) \) implies that \( x \in (A \cup B) \cup C \) and so \( (A \cup (B \cup C)) \subset ((A \cup B) \cup C) \). Since we have subset inclusion in both directions, we conclude set equality.

- **Associativity (1B):** \( x \in (A \cap B) \cap C \) if and only if \( x \in A \cap B \) and \( x \in C \) if and only if \( x \in A \) and \( x \in B \), and \( x \in C \) if and only if \( x \in A \) and \( x \in B \) and \( x \in C \) if and only if \( x \in A \) and \( x \in B \cap C \) if and only if \( x \in A \cap (B \cap C) \). This is sufficient (by Axiom of Extensionality) to show that the two sets are equal.

- **Idempotence (3A):** \( x \in (A \cup A) \) if and only if \( x \in A \) or \( x \in A \) if and only if \( x \in A \). Extensionality gives the result.

- **Idempotence (3B):** Suppose \( x \in A \cap A \). Then \( x \in A \) and \( x \in A \). Hence, \( x \in A \). So \( A \cap A \subset A \). Now suppose that \( x \in A \). Then \( x \in A \) and \( x \in A \). So \( x \in A \cap A \) and \( A \cap A \subset A \).

- **DeMorgan’s (5A):** Suppose \( x \in \neg (A \cap B) \). So \( x \) is not in \( A \cap B \). Case 1 - \( x \) is not in \( A \). Then \( x \in \neg A \), so \( x \in \neg A \cup \text{(anything)} \) and in particular \( x \in \neg A \cap \neg B \). Case 2 - \( x \) is not in \( B \). Then \( x \in \neg B \), so \( x \in \text{(anything)} \cup \neg B \) and in particular \( x \in \neg A \cup \neg B \). So in all cases, \( x \in \neg (A \cap B) \) implies that \( x \in \neg A \cup \neg B \). Hence, \( \neg (A \cap B) \subset \neg A \cup \neg B \). Now suppose that \( x \in \neg A \cup \neg B \). Then either \( x \in \neg A \) or \( x \in \neg B \) (or both). If \( x \in \neg A \) then \( x \) is not in \( A \) so \( x \) is not in \( A \cap \text{(anything)} \) and in particular, \( x \in \neg (A \cap B) \). Similarly, if \( x \in \neg B \) then \( x \) is not in \( B \) so \( x \) is not in \( \text{(anything)} \cap B \) and in particular, \( x \in \neg (A \cap B) \). Therefore, \( x \in \neg A \cup \neg B \) implies that \( x \in \neg (A \cap B) \) and so \( \neg A \cup \neg B \subset \neg (A \cap B) \). Subset inclusion in both directions implies set equality.

- **Empty Set (7A):** \( x \in A \cup \emptyset \) if and only if \( x \in A \) or \( x \in \emptyset \) if and only if \( x \in A \) (since \( \emptyset \) has no members).

- **Empty Set (7B):** \( x \in A \cap \emptyset \) if and only if \( x \in A \) and \( x \in \emptyset \) if and only if \( x \in A \) (since there can be no \( x \in \emptyset \)) if and only if \( x \in \emptyset \).

- **Double Negation (9):** Suppose \( x \in A \). Then \( x \) is not not in \( A \), so \( x \) is not in \( \neg \neg A \). Hence, \( x \in \neg \neg A \) and \( A \subset \neg \neg A \). Suppose \( x \in \neg \neg A \). Then \( x \) is not in \( A \), so \( x \) is not not in \( A \). Hence, \( x \in A \) and \( \neg \neg A \subset A \). Subset inclusion in both directions implies set equality.

**Exercise 4.** Suppose that \( R, S, T \) are arbitrary binary relations on a set \( X \), \( I = \{ (x, x) : x \in X \} \) is the identity relation, \( U = \{ (x, y) : x, y \in X \} \) is universal relation. Prove that each of the Boolean laws for sets in the last section holds for \( R, S, T \) (assume \( \neg R \) denotes \( U - R \)).

**Solution 4.** To show that two relations \( R, S \) on \( X \) are equal, it suffices to show that for any arbitrary ordered pair \( \langle a, b \rangle \), \( aRb \) if and only if \( aSb \).

- **Commutativity (2A):** WTS \( (R \cup S) = (S \cup R) \). \( a(R \cup S)b \) if and only if \( aRb \) and \( aSb \) if and only if \( aSb \) and \( aRb \) if and only if \( a(S \cup R)b \)

- **Commutativity (2B):** WTS \( (R \cap S) = (S \cap R) \). \( a(R \cap S)b \) if and only if \( aRb \) or \( aSb \) if and only if \( aRb \) or \( aSb \) if and only if \( a(S \cap R)b \)

- **Distributivity (4A):** WTS \( R \cup (S \cap T) = (R \cup S) \cap (R \cup T) \). Suppose \( \langle a, b \rangle \in R \cup (S \cap T) \). Then \( aRb \) or \( (aSb \text{ and } aTb) \). Case 1 - \( aRb \). Then \( a(R \cup \text{(anything)})b \). So both \( a(R \cup S)b \) and \( a(R \cup T)b \). Hence \( a((R \cup S) \cap (R \cup T))b \). Case 2 - \( aSb \) and \( aTb \). Then \( a((\text{(anything)} \cup S)b \) and \( a((\text{anything}) \cup T)b \). In particular, \( a(R \cup S)b \) and \( a(R \cup T)b \). So in both cases we have that if \( \langle a, b \rangle \in R \cup (S \cap T) \) then \( \langle a, b \rangle \in (R \cup S) \cap (R \cup T) \). Conversely, suppose that \( \langle a, b \rangle \in (R \cup S) \cap (R \cup T) \). Then \( a(R \cup S)b \) and \( a(R \cup T)b \). Case 1 - \( aRb \). Then \( a(R \cup \text{(anything)})b \) and in particular \( a(R \cup (S \cap T))b \). Case 2 - \( aSb \) and \( aTb \) (think about
why these two cases are exhaustive!). Then \( a(S \cap T)b \) and also \( a((\text{anything}) \cup (S \cap T))b \). In particular, \( a(R \cup (S \cap T))b \). Therefore, if \( \langle a, b \rangle \in (R \cup S) \cap (R \cup T) \) then \( \langle a, b \rangle \in R \cup (S \cap T) \). We are now done proving the biconditional.

- Negation (6A): WTS \( R \cup (\neg R) = U \). \( a(R \cup (\neg R))b \) if and only if \( aRb \) or \( a(U - R)b \) if and only if \( aRb \) or \( a, b \in X \) and not \( aRb \) if and only if always (by Law of Excluded Middle) if and only if \( aUb \).
- Negation (6B): WTS \( R \cap (\neg R) = \emptyset \). \( aR \cap (\neg R)b \) if and only if \( aRb \) and \( a(U - R)b \) if and only if \( aRb \) and not \( aRb \) if and only if IMPOSSIBLE! if and only if \( a\emptyset b \).
- Absorption (8A): WTS \( R \cup U = U \). \( a(R \cup U)b \) if and only if \( aRb \) or \( aUb \) if and only if always true since \( aUb \) always holds if and only if \( aUb \).
- Absorption (8B): WTS \( R \cap U = R \). \( a(R \cap U)b \) if and only if \( aRb \) and \( aUb \) if and only if \( aRb \) (since \( aUb \) always holds).

**Exercise 5.** Suppose that \( f \) and \( g \) are maps

(a) Prove that \( f \circ g = f \times g \)
(b) Prove that if \( f \) is 1-1 then \( \tilde{f} \) is a function. What is its domain and range?
(c) Prove that if \( f \) is a bijection then so is \( \tilde{f} \).
(d) Prove that if \( f \) and \( g \) are bijections then so is \( f \circ g \).
(e) Prove that if \( f \) is an injection then \( f^{-1} = \tilde{f} \upharpoonright \operatorname{rg}(f) \) is a function such that \( f^{-1} \circ f(x) = x \) for every \( x \in \operatorname{dom}(f) \) and \( f \circ f^{-1}(y) = y \) for every \( y \in \operatorname{rg}(f) \).

**Solution 5.**

(a) Since binary relations are sets, to show that two binary relations \( R, S \) are equal, it is necessary and sufficient to show that for any ordered pair \( \langle a, b \rangle \), \( aRb \) iff \( aSb \). By definition of composition of relations, \( af \times gb \) if and only if there exists \( c \) such that \( acg \) and \( cfb \) if and only if there exist \( c \) such that \( g(a) = c \) and \( f(c) = b \) if and only if \( (f \circ g)(a) = b \) (by definition of composition of functions).

(b) Suppose that \( f \) is injective. Consider \( \tilde{f} \), the converse of \( f \). This is a set of ordered pairs. To show that \( \tilde{f} \) is a function we must show that for every \( a, b_1, b_2 \), \( a\tilde{f}b_1 \) and \( a\tilde{f}b_2 \) imply \( b_1 = b_2 \). So, suppose that \( a\tilde{f}b_1 \) and \( a\tilde{f}b_2 \). The definition of converse gives that \( b_1fa \) and \( b_2fa \), i.e. \( f(b_1) = f(b_2) = a \). Injectivity then gives that \( b_1 = b_2 \), as required for \( \tilde{f} \) to be a map. The domain of \( \tilde{f} \) is the first projection of \( \tilde{f} \) which is the second projection of \( f \). Hence, the domain of \( \tilde{f} \) is the range of \( f \). Analogously, the range of \( \tilde{f} \) is the domain of \( f \). So, \( \tilde{f} : \operatorname{rg}(f) \to A \).

(c) Suppose \( f \) is bijective. Then \( \tilde{f} : B \to A \) (by (a) and since \( f \) bijective means \( \operatorname{rg}(f) = B \)). For injectivity: Suppose \( \tilde{f}(b_1) = \tilde{f}(b_2) = a \) for some \( a, b_1, b_2 \). Then, \( f(a) = b_1 \) and \( f(a) = b_2 \). Since \( f \) is a function, this implies that \( b_1 = b_2 \). For surjectivity: Let \( a \in A \). Since \( A = \operatorname{dom}(f) \) and \( B = \operatorname{rg}(f) \) there is some \( b \in B \) such that \( f(a) = b \). Hence, \( \tilde{f}(b) = a \) and so for every element in \( A \) there is some element in \( B \) mapped to it by \( \tilde{f} \).

(d) Suppose that \( f : B \to C \) and \( g : A \to B \) are bijections. Consider \( f \circ g \). Injectivity: Suppose \( f(g(a_1)) = f(g(a_2)) = c \). Then by injectivity of \( f \), \( g(a_1) = g(a_2) \). Injectivity of \( g \) gives that \( a_1 = a_2 \). Surjectivity: Let \( c \in C \). Since \( f \) is onto, there is \( b \in B \) such that \( f(b) = c \). But, since \( g \) is onto, there is \( a \in A \) such that \( g(a) = b \). So, \( (f \circ g)(a) = f(g(a)) = f(b) = c \), as required.
(e) By (a) we have that since $f$ is an injection $\bar{f}$ is a function, and hence any restriction of $\bar{f}$ is a function. Let $x \in \text{dom}(f)$. Then $f(x) \in \text{rg}(f) = \text{dom}(f^{-1})$. So, $f^{-1} \circ f(x) = (\bar{f} \circ f)(x) = \bar{f}(f(x)) = x$ (Why? Let $f(x) = y$. Then $\bar{f}(y) = x$ by definition of converse. Hence, $\bar{f}(f(x)) = \bar{f}(y) = x$). Now let $y \in \text{rg}(f) = \text{dom}(f^{-1})$. Then $f \circ f^{-1}(y) = f(f^{-1}(y))$ by similar arguments.