1 Sets

“A set is a collection of elements”. We can describe a set by $A = \{a, b, c, d, \cdots \}$ or $A = \{x \mid x \text{ is blah blah blah} \}$.

- $x \in A$ if an element $x$ belongs to a set $A$. ($x \notin A$ if an element $x$ does not belong to a set $A$. )
- $A \subset B$ if a set $A$ is a subset of a set $B$, i.e. all elements of $A$ belong to $B$.
- $A \subseteq B$ if $A$ is a proper subset of $B$, i.e. $A \subset B$ and $A \neq B$.
- The complement $A^c$ of a subset $A$ of $B$ is defined by $A^c := \{b \in B \mid b \notin A \}$.
- The empty set $\emptyset$ is defined to be the set with no elements.
- The power-set $\mathcal{P}(A)$ of a set $A$ is defined to be the set of all subsets of $A$.

The natural maps $\pi_1 : A \times B \to A$, $(a, b) \mapsto a$ and $\pi_2 : A \times B \to B$, $(a, b) \mapsto b$ are called the projections (to the 1st and 2nd factors).

The union, the intersection and the difference of two sets $A$ and $B$ are the sets defined as follows:

- Union: $A \cup B := \{x \mid x \in A \text{ or } x \in B \}$.
- Intersection: $A \cap B := \{x \mid x \in A \text{ and } x \in B \}$.
- Difference: $A - B := \{x \mid x \in A \text{ and } x \notin B \}$.

We will use the symbol $:= \text{ to define a set.}$

See page 11 of [Mun] and check the formulas there. We can also define arbitrary unions and intersections over any set of sets: let $I$ be a set of sets, then

$$\bigcup_{A \in I} A := \{x \mid x \in A \text{ for at least one } A \in I \} \quad \text{and} \quad \bigcap_{A \in I} A := \{x \mid x \in A \text{ for all } A \in I \}.$$

Caution: A collection of sets is not a set in general. For example, if we regard the collection of all sets as a set, we will get into theoretical troubles. In more general framework, such a collection is named a proper class. See for example “Class (set theory)” in wikipedia. As long as we consider collections of subsets of a set $A$, it’s fine. They are sets, and indeed, subsets of the power-set $\mathcal{P}(A)$. 

2 Maps (= Functions)

A map $f$ from a set $A$ to a set $B$ is an assignment of an element of $B$ to each element of $A$, denoted by

$$f : A \to B, \ a \mapsto f(a) \quad \text{where} \ a \in A \text{ and } b := f(a) \in B.$$  

- The composition $g \circ f : A \to C$ of maps $f : A \to B$ and $g : B \to C$ is defined by $(g \circ f)(a) := g(f(a))$.
- The image of a map $f : A \to B$ is the subset of $B$ given by
  $$\text{Im} f := f(A) = \{ b \in B \mid \exists a \in A \text{ such that } b = f(a) \}.$$  
- Let $f : A \to B$ be a map. The inverse image (or preimage) $f^{-1}(B')$ of a subset $B' \subset B$ is the subset of $A$ given by
  $$f^{-1}(B') := \{ a \in A \mid f(a) \in B' \}.$$  
- The restriction of a map $f : A \to B$ to a subset $A' \subset A$ is the map $f|_{A'} : A' \to B$ given by
  $$f|_{A'}(a') := f(a') \quad \text{for all } a' \in A'.$$

The injectivity, surjectivity and bijectivity of a map $f : A \to B$ is defined as follows:

- $f$ is injective if no two distinct elements of $A$ go to the same element in $B$ under $f$, i.e. $f(a) = f(a')$ implies $a = a'$
- $f$ is surjective if the image of $f$ coincides with $B$.
- $f$ is bijective (a one to one correspondence) of $f$ is injective and surjective.

If $f : A \to B$ is bijective, then there is a map $f^{-1} : B \to A$ called the inverse of $f$. This map $f^{-1}$ sends $b \in B$ to the unique element $a \in A$ such that $f(a) = b$.

2.1 Compatibility between maps and set theoretical operations $\subset, \cup, \cap$ of subsets

Let $f : X \to Y$ be a map and $A_1, A_2 \subset X$ and $B_1, B_2 \subset Y$ subsets. Then $f^{-1}$ preserves unions, intersections, inclusions:

(a) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
(b) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
(c) $B_1 \subset B_2$ implies $f^{-1}(B_1) \subset f^{-1}(B_2)$

$f$ preserves unions and intersections only:

(d) $f(A_1 \cup A_2) \supset f(A_1) \cup f(A_2)$.
(e) $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$; the equality holds if $f$ is injective.
(f) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

(a), (b), (d) and (e) hold for arbitrary unions and intersections.
3 Relations

A relation $\sim_R$ on a set $A$ is given by a subset $R \subset A \times A$. If $(x, y) \in R$, then $x \sim_R y$ ($x$ is related to $y$).

(I) An equivalence relation $\sim$ on $A$ is a relation such that

1. (Reflexivity) $x \sim x$
2. (Symmetry) $x \sim y$ implies $y \sim x$
3. (Transitivity) $x \sim y$ and $y \sim z$ implies $x \sim z$

- For an equivalence relation $\sim$ on $A$ and an element $x \in A$, the equivalence class $[x]$ is a subset of $A$ given by

$$[x] := \{y \in A \mid x \sim y\} \subset A.$$  

By the reflexivity $x \sim x$, we have $x \in [x]$.

Lemma 3.1.

1. $[x] = [y]$ if and only if $x \sim y$.
2. $[x] \cap [y] = \emptyset$ if and only if $x \not\sim y$.
3. Let $E_1$ and $E_2$ be equivalence classes, then either $E_1 = E_2$ or $E_1 \cap E_2 = \emptyset$.

Remark 3.2. An equivalence relation on $A$ defines a surjection $\pi : A \to E$ where $E$ is the set of all equivalence classes. On the other hand, if we have a surjection $\pi : A \to \mathcal{E}$ to some set $\mathcal{E}$, then we can define an equivalence relation on $A$ by saying, for each $E_1 \in \mathcal{E}$, all points in $\pi^{-1}(E_1)$ are equivalent to each other. Thus there are one-to-one correspondence between surjective maps and equivalence classes.

(II) An order relation (or a simple order, or a linear order) $<$ on $A$ is a relation such that

1. (Comparability) If $x \neq y$, then $x < y$ or $y < x$.
2. (Non-reflexivity) If $x < y$, then $x \neq y$.
3. (Transitivity) If $x < y$ and $y < z$, then $x < z$.

- If $<$ is an order relation, $x \leq y$ means that either $x < y$ or $x = y$.

References