Degree of maps from $S^1$ to $S^1$

We want to define a homotopy invariant for continuous maps $S^1 \to S^1$ to classify them. Follow the instructions and answer the questions (1) – (8).

Let $S^1 := \{e^{2\pi i \theta} \in \mathbb{C} \}$. Let $x_0 \in S^1$ and let $\alpha$ be a path from $1 \in S^1$ to $x_0$. Let $\gamma$ be a generator of $\pi_1(S^1, 1)$.

(1) Show that $\hat{\alpha}(\gamma)$ is a generator of $\pi_1(S^1, x_0)$. See Appendix *1.

(2) Show that $\hat{\alpha}(\gamma)$ depends only on $x_0$ (not on paths $\alpha$).

By (2), it is OK to write $\gamma_{x_0} := \hat{\alpha}(\gamma)$ for any path $\alpha$ from $1$ to $x_0$. Now let $h : S^1 \to S^1$ be a continuous map. Let $x_0 \in S^1$ and let $x_1 := h(x_0)$. Define degree of $h$ to be an integer $d$ such that $h_*(\gamma_{x_0}) = (\gamma_{x_1})^d$.

It is well-defined because, by (1), $\gamma_{x_1}$ is a generator of $\pi_1(S^1, x_1)$.

(3) Show that $d$ is independent of the choice of $x_0$.

(4) Show that $d$ is independent of the choice of $\gamma$.

By (3) and (4), we have defined the degree of a map $h$, which is independent of all choices. Now we consider the properties of this degree:

(5) Show that if $h, k : S^1 \to S^1$ are homotopic, they have the same degree.

(6) Show that $\deg h \circ k = \deg h \cdot \deg k$.

(7) Compute the degree of the map $h(z) = z^n$ where $n \in \mathbb{Z}$.

(8) (Optional) Show that if $h, k : S^1 \to S^1$ have the same degree, then they are homotopic.

(5) says the degree is a homotopy invariant, i.e. if $h, k$ have the different degrees, they can not be homotopic to each other. Together with (7) and (8), it classifies all homotopy equivalence classes of maps $S^1 \to S^1$.

(6) says associating degrees have a certain algebraic structure.

Appendix

*1 An infinite cyclic group $G$ is a group isomorphic to $(\mathbb{Z}, +)$. We say $g \in G$ is a generator, if $G = \{g^n \mid n \in \mathbb{Z} \}$. If $g$ is a generator, then $g^{-1}$ is also a generator and any generator is either $g$ or $g^{-1}$. 