(1) Show that $X$ is a Hausdorff space if and only if the diagonal $\Delta := \{(x, x) \mid x \in X\} \subset X \times X$ is closed with respect to the product topology.

(2) Find all points that the sequence $\{x_n = 1/n \mid \mathbb{Z} > 0\}$ converges to with respect to the following topology of $\mathbb{R}$. Justify your answer.
   (a) Standard Topology
   (b) Finite Complement Topology
   (c) Discrete Topology
   (d) Lower Limit Topology

(3) Let $X$ and $Y$ be topological spaces. Prove that $f : X \to Y$ is continuous if and only if for every subset $A$ of $X$, we have $f(A) \subset f(A)$.

(4) Define a map $f : \mathbb{R} \to \mathbb{R}$ by
   
   $x \mapsto \begin{cases} 
   |x| & \text{if } x \text{ is rational} \\
   -|x| & \text{if } x \text{ is irrational}.
   \end{cases}$

   Show that $f$ is continuous at $x = 0$ but not continuous at other points.

(5) Let $X$ and $Y$ be sets. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections to the first and the second factors.
   (a) (Set Theory) For a given set $Z$ with maps $f_1 : Z \to X$ and $f_2 : Z \to Y$, find a map $g : Z \to X \times Y$ such that $f_1 = \pi_1 \circ g$ and $f_2 = \pi_2 \circ g$. Show that such $g$ is unique.
   (b) Suppose $X$ and $Y$ are topological spaces. Show that $\pi_1$ and $\pi_2$ are continuous maps with respect to the product topology $T_{\text{prod}}$ on $X \times Y$. Show that any topology $T$ on $X \times Y$ such that $\pi_1$ and $\pi_2$ are continuous, must be finer than the product topology.
   (c) Suppose $X$ and $Y$ are topological spaces. For a given topological space $Z$ with continuous maps $f_1 : Z \to X$ and $f_2 : Z \to Y$, show that the map $g$ you found in (a) is continuous with respect to the product topology on $X \times Y$.
   (d) Explain why there is no finer topology on $X \times Y$ than the product topology such that (c) holds.

(6) Show that the open interval $(-\pi/2, \pi/2)$ of $\mathbb{R}$ with the subspace topology is homeomorphic to $\mathbb{R}$. Show that any open interval is homeomorphic to $\mathbb{R}$.

(7) Suppose that $X, Y, Z$ are topological spaces. Let $f : X \to Y$ and $g : Y \to Z$ be maps of sets. Prove or disprove the following statement:
   (a) If $f : X \to Y$ is continuous and the composition map $g \circ f : X \to Z$ is continuous, then $g : Y \to Z$ is continuous.
   (b) If $g : Y \to Z$ is continuous and the composition map $g \circ f : X \to Z$ is continuous, then $f : X \to Y$ is continuous.

References