Note: Each problem is worth 14 points except numbers 5 and 6 which are 15 points.

1. Compute \( \int\int_D x^2 + y^2 \, dA \) where \( D \) is the region in the second quadrant between the circles \( x^2 + y^2 = 1 \) and \( x^2 + y^2 = 4 \).

Solution. In polar coordinates the region \( D \) is given by the inequalities \( 1 \leq r \leq 2 \) and \( \pi/2 \leq \theta \leq \pi \). The integral then becomes

\[
\int_{\pi/2}^{\pi} \int_{1}^{2} r^2 \cos^2 \theta \, r \, dr \, d\theta = \int_{1}^{2} r^3 \cos^2 \theta \, dr \, \left[ \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_{\pi/2}^{\pi} = \frac{3}{8} \pi - \frac{3}{8} \pi = \frac{3\pi}{8}
\]

2. Let \( D \) be the region in the first quadrant bounded by the curves \( y = x^\alpha \) and \( y = x^\beta \) for \( 0 < \beta < \alpha \). Compute \( \int\int_D xy \, dA \) in two ways, integrating in the order \( dx \, dy \) and in the order \( dy \, dx \), and verify that the answer is the same in both cases.

Solution. The graphs of \( x^\alpha \) and \( x^\beta \) intersect at the points \((0,0)\) and \((1,1)\). Between these two points the graph of \( x^\beta \) is above the graph of \( x^\alpha \). Integrating in the order \( dy \, dx \) we then have

\[
\int_{0}^{1} \int_{x^\alpha}^{x^\beta} xy \, dy \, dx = \int_{0}^{1} \left( \frac{1}{2} xy^2 \right)_{x^\alpha}^{x^\beta} \, dx = \frac{1}{2} \int_{0}^{1} x^{2\beta+1} - x^{2\alpha+1} \, dx = \frac{1}{2} \left( \frac{x^{2\beta+2}}{2\beta+2} - \frac{x^{2\alpha+2}}{2\alpha+2} \right) \bigg|_{0}^{1} = \frac{1}{2} \left( \frac{1}{2\beta+2} - \frac{1}{2\alpha+2} \right) = \frac{\alpha - \beta}{4(\alpha + 1)(\beta + 1)}
\]

If we change to the order \( dx \, dy \) then the integral becomes

\[
\int_{0}^{1} \int_{y^{1/\beta}}^{y^{1/\alpha}} xy \, dx \, dy = \int_{0}^{1} \frac{1}{2} x^2 y \left|_{y^{1/\beta}}^{y^{1/\alpha}} \right. \, dy = \frac{1}{2} \int_{0}^{1} y^{(2/\alpha)+1} - y^{(2/\beta)+1} \, dy
\]

\[
= \frac{1}{2} \left( \frac{y^{(2/\alpha)+2}}{(2/\alpha) + 2} - \frac{y^{(2/\beta)+2}}{(2/\beta) + 2} \right) \bigg|_{0}^{1} = \frac{1}{2} \left( \frac{1}{2/\alpha + 2} - \frac{1}{2/\beta + 2} \right)
\]

\[
= \frac{1}{2} \left( \frac{\alpha}{2\alpha + 2} - \frac{\beta}{2\beta + 2} \right) = \frac{\alpha - \beta}{4(\alpha + 1)(\beta + 1)}
\]

So we get the same answer in both cases.
3. Consider an integral \( \int_0^2 \int_0^{2-x} \int_0^{4-2y} f(x, y, z) \, dz \, dy \, dx \).

(a) Sketch the region of integration.

Solution. We can determine the region from the limits of integration. For the \( dz \) integration we see that \( z \) goes from \( z = 0 \) (the \( xy \)-plane) to \( z = 4 - 2y \) (a plane sloping down to the right). For the \( dy \) and \( dx \) integrations we are looking at the shadow of the region in the \( xy \) plane, a triangle in the first quadrant cut off by the line \( y = 2 - x \). The 3-dimensional region of integration lies above this triangle and below the plane \( z = 4 - 2y \), so it has the shape shown in the figure, sort of a pyramid lying on its side.

(b) Set up the integral in the order \( dx \, dy \, dz \) (but do not attempt to evaluate the integral since we have not specified what the function \( f(x, y, z) \) is).

Solution. For the order \( dx \, dy \, dz \) we are projecting to the shadow in the \( yz \)-plane, which is the triangle in the first quadrant of the \( yz \)-plane cut off by the line \( z = 4 - 2y \), or \( y = 2 - (z/2) \). For the initial \( dx \) integration we have \( x \) going from \( x = 0 \) (the \( yz \)-plane) to \( x = 2 - y \) (a vertical plane). The integral is then

\[
\int_0^4 \int_0^{2-(z/2)} \int_0^{2-y} f(x, y, z) \, dx \, dy \, dz.
\]

4. Compute the volume of the region bounded by the \( xy \)-plane and the two surfaces \( z = 1 - x^2 \) and \( z = 1 - y^2 \) (these are parabolic cylinders). Hint: break the region up into several symmetric pieces of equal volume.

Solution. The region is shown in the figure on the left below. It is tent-shaped, and the projection of the region to the \( xy \)-plane is the square \(-1 \leq x \leq 1\) and \(-1 \leq y \leq 1\).

The region is symmetric under reflection across the \( yz \)-plane and the \( xz \)-plane. It is also symmetric under reflection across the vertical planes \( x = \pm y \). We can compute the volume
by computing the volume of the piece shown in the right half of the figure and multiplying this by 8. For this piece the curved upper surface is part of the surface \( z = 1 - x^2 \), so the volume of this piece is

\[
\int_0^1 \int_0^x \int_0^{1-x^2} 1 \, dz \, dy \, dx
= \int_0^1 \int_0^x \left( z \bigg|_0^{1-x^2} \right) \, dy \, dx
= \int_0^1 (1 - x^2) \left. y \right|_0^x \, dx
= \int_0^1 (x - x^3) \, dx
= \left( \frac{x^2}{2} - \frac{x^4}{4} \right) \bigg|_0^1
= \frac{1}{4}
\]

Therefore the volume of the region is equal to \( 8 \times \frac{1}{4} = 2 \).

There are other ways to compute the volume, using different orders for \( dx \), \( dy \), and \( dz \). For example:

\[
4 \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{\sqrt{1-z}} \, dx \, dy \, dz
or
4 \int_0^1 \int_0^{1-y^2} \int_0^{\sqrt{1-z}} \, dx \, dz \, dy
\]

5. Find the center of gravity of the constant-density solid cone consisting of all points \((x, y, z)\) satisfying the inequalities \( \sqrt{x^2 + y^2} \leq z \leq a \), where \( a \) is an arbitrary constant.

**Solution.** We use cylindrical coordinates. The inequalities \( \sqrt{x^2 + y^2} \leq z \leq a \) then become \( r \leq z \leq a \).

The volume of the cone is given by the integral

\[
\iiint_D 1 \, dV = \int_0^{2\pi} \int_0^a \int_r^a r \, dz \, dr \, d\theta
= \int_0^{2\pi} \int_0^a rz \bigg|_r^a \, dr \, d\theta
= \int_0^{2\pi} \int_0^a (ar - r^2) \, dr \, d\theta
= \int_0^{2\pi} \left( \frac{ar^2}{2} - \frac{r^3}{3} \right) \bigg|_0^a \, d\theta
= \int_0^{2\pi} \frac{a^3}{6} \, d\theta
= \frac{\pi a^3}{3}
\]

Using the rotational symmetry of the cone we can see that the center of gravity lies on the \( z \)-axis so its \( x \) and \( y \) coordinates are 0. In order to compute the \( z \) coordinate we need the integral

\[
\iiint_D z \, dV = \int_0^{2\pi} \int_0^a \int_r^a rz \, dz \, dr \, d\theta
= \int_0^{2\pi} \int_0^a \left( \frac{r^2}{2} \right) \bigg|_r^a \, dz \, dr \, d\theta
= \frac{1}{2} \int_0^{2\pi} \left( \frac{a^2r^2}{2} - \frac{r^4}{4} \right) \bigg|_0^a \, d\theta
= \frac{1}{2} \int_0^{2\pi} \frac{a^4}{4} \, d\theta
= \frac{\pi a^4}{8}
\]

Finally, the \( z \) coordinate of the center of mass is

\[
\bar{z} = \frac{\iiint_D z \, dV}{\iiint_D 1 \, dV} = \frac{\pi a^4/4}{\pi a^3/3} = \frac{3a}{4}
\]
6. Find the volume of the region that lies inside the sphere \( x^2 + y^2 + z^2 = 2z \) and outside the sphere \( x^2 + y^2 + z^2 = 2 \).

Solution. The sphere \( x^2 + y^2 + z^2 = 2 \) has radius \( \sqrt{2} \) with center at the origin. The equation for the other sphere can be rewritten as \( x^2 + y^2 + (z - 1)^2 = 1 \) so this sphere has radius 1 and center at \((0, 0, 1)\). In cross section the two spheres look like the circles in the figure at the right. The region we want is the shaded region.

We use spherical coordinates to compute the volume. In spherical coordinates the lower sphere has equation \( \rho = \sqrt{2} \) and the upper sphere has equation \( \rho^2 = 2z = 2\rho \cos \varphi \), or \( \rho = 2 \cos \varphi \). These two spheres intersect in the circle where the plane \( z = 1 \) intersects the two spheres, so \( \varphi = \pi/4 \) on this circle. Therefore the region can be described in spherical coordinates as

\[
\sqrt{2} \leq \rho \leq 2 \cos \varphi \quad 0 \leq \varphi \leq \pi/4 \quad 0 \leq \theta \leq 2\pi
\]

The volume of the region is given by the integral

\[
\int_0^{2\pi} \int_0^{\pi/4} \int_{\sqrt{2}}^{2 \cos \varphi} \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} \int_0^{\pi/4} \frac{\rho^3}{3} \sin \varphi \bigg|_{\sqrt{2}}^{2 \cos \varphi} \, d\varphi \, d\theta
\]

\[
= \frac{1}{3} \int_0^{2\pi} \, d\theta \int_0^{\pi/4} (8 \cos^3 \varphi \sin \varphi - 2\sqrt{2} \sin \varphi) \, d\varphi
\]

\[
= \frac{2\pi}{3} \left( -2 \cos^4 \varphi + 2\sqrt{2} \cos \varphi \right) \Big|_{\pi/4}^{\pi/4} = \frac{\pi}{3} (7 - 4\sqrt{2})
\]

Here we used the substitution \( u = \cos \varphi \) to compute the \( d\varphi \) integral.

7. Let \( D \) be the region in \( \mathbb{R}^2 \) defined by the inequalities \( y \leq x \leq 2y \) and \( 3 \leq x + y \leq 4 \). Compute \( \iint_D \frac{1}{xy} \, dx \, dy \) by using a change of variables to convert \( D \) into a square. Hint: Rewrite the inequalities \( y \leq x \leq 2y \) as \( 1 \leq x/y \leq 2 \).

Solution. We want to simplify the region \( D \) defined by the inequalities \( 3 \leq x + y \leq 4 \) and \( 1 \leq x/y \leq 2 \), and the easiest way to do this is to choose new variables \( u = x/y \) and \( v = x + y \) so that the inequalities become \( 1 \leq u \leq 2 \) and \( 3 \leq v \leq 4 \), which define a square. To carry out the change of variables from \( x \) and \( y \) to \( u \) and \( v \) we will need to express \( x \) and \( y \) in terms of \( u, v \), i.e., we need to solve for \( x \) and \( y \) in the system

\[
x/y = u \quad x + y = v.
\]
Solving the first equation for \( x \) gives \( x = uy \) and substituting this in the second equation gives \( uy + y = v \), which can be solved for \( y \) to get \( y = \frac{v}{u+1} \). Then the earlier equation \( x = uy \) gives \( x = \frac{uv}{u+1} \). Thus we have \( x = \frac{uv}{u+1} \) and \( y = \frac{v}{u+1} \).

Next we need to compute the Jacobian determinant:

\[
\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix} = \det \begin{vmatrix}
\frac{u}{(u+1)^2} & \frac{w}{u+1} \\
\frac{v}{(u+1)^2} & \frac{1}{u+1}
\end{vmatrix} = \frac{v + uv}{(u + 1)^3} = \frac{v}{(u + 1)^2}.
\]

The change of variables formula gives

\[
\iint_D xy \, dA = \iint_{D'} x(u, v)y(u, v) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dA = \\
\iint_{D'} \frac{uv}{u+1} \frac{v}{(u+1)^2} \, dA = \iint_{D'} \frac{1}{uv} \, dA,
\]

where \( D' \) is the region in the \( uv \)-plane corresponding to \( D \), so \( D' \) is the square \( 1 \leq u \leq 2 \), \( 3 \leq v \leq 4 \). Thus we get

\[
\int_1^2 \int_3^4 \frac{1}{uv} \, dv \, du = \int_1^2 \frac{1}{u} \, du \int_3^4 \frac{1}{v} \, dv = \left( \ln u \right)_1^2 \left( \ln v \right)_3^4 = \ln 2(\ln 4 - \ln 3)
\]