Math 2220 Final
May 11, 2009
Name: ________________________
Instructor: ____________________
Lecture time: __________________
TA: __________________________
Section time: __________________

INSTRUCTIONS — READ THIS NOW
• This test has 10 problems on 16 pages worth a total of 200 points.
  Look over your test package right now. If you find that any pages are
  missing or find any other problems please ask a proctor for another test
  booklet.
• Write your name, your instructor’s name, the time your lecture meets,
  your TA’s name, and the time you section meets right now.
• Show your work. To receive full credit, your answers must be neatly
  written and logically organized. If you need more space, write on the
  back side of the preceding sheet, but be sure to clearly label your work.
• This is a 150 minute test. You are not allowed to use a calculator
  or any other aids except for one page (front and back) of notes and
  formulas. You DO NOT need to SIMPLIFY your answers.

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Problem 1: (20 points)

Let $V$ be the set of all points $(x, y, z)$ satisfying $0 \leq z \leq 2$ and $x^2/9 + y^2/16 \leq 1$. Evaluate

$$\iiint_V x \, dx \, dy \, dz$$

using the change of variables

$$x = 3r \cos(\theta)$$
$$y = 4r \sin(\theta)$$
$$z = z$$

Solution. Let $U = \left\{ (r, \theta, z) \left| 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 2 \right. \right\}$, and $\Phi: U \to V$ be given by $\Phi(r, \theta, z) = (3r \cos \theta, 4r \sin \theta, z)$. Then $\Phi$ maps $U$ one-to-one onto $V$ and

$$\iiint_V x \, dx \, dy \, dz = \int_0^2 \int_0^1 \int_0^{2\pi} 3r \cos \theta \left| \frac{\partial(x, y, z)}{\partial(r, \theta, z)} \right| \, d\theta \, dr \, dz$$
$$= \int_0^2 \int_0^1 \int_0^{2\pi} 36r^2 \cos \theta \, d\theta \, dr \, dz$$
$$= 0.$$

This makes sense since the average value of $x$ over the region $V$ should be zero. \qed
Problem 2: (20 points)

Sketch the region of integration and evaluate \( \int_0^1 \int_{2y}^2 (x^2 + 1)^{1/3} \, dx \, dy. \)

Solution. The region is the triangle with vertices \((0, 0), (2, 0),\) and \((2, 1)\). Thus

\[
\int_0^1 \int_{2y}^2 (x^2 + 1)^{1/3} \, dx \, dy = \int_0^2 \int_0^{x/2} (x^2 + 1)^{1/3} \, dy \, dx \\
= \int_0^2 \frac{x}{2} (x^2 + 1)^{1/3} \, dx \\
= \frac{3}{16} (x^4 + 1)^{4/3} \bigg|_0^2 \\
= \frac{3}{16} (5^{4/3} - 1).
\]

\(\Box\)
Problem 3: (15 points)

The curves $c_1$ and $c_2$ defined by

\[
\begin{align*}
c_1(t) &= (t, 2t, t^2) \\
c_2(t) &= (t^2, 1 - t, 2 - t^2)
\end{align*}
\]

intersect at the point $(1, 2, 1)$. Find the plane tangent to both curves at that point.

Solution. Note that $c_1(1) = (1, 2, 1) = c_2(-1)$. To find the tangent plane at $(1, 2, 1)$, we need to find a vector which is normal to both curves at that point. This is accomplished by taking the cross product of the tangent vectors at $(1, 2, 1)$. Since $c_1'(1) = (1, 2, 2t)$ and $c_2'(t) = (2t, -1, -2t)$, we have that the tangent vectors at $(1, 2, 1)$ are $c_1'(1) = (1, 2, 2)$ and $c_2'(-1) = (-2, -1, 2)$. Thus, the normal vector is $(1, 2, 2) \times (-2, -1, 2) = (6, -6, 3)$, and the equation of the tangent plane is $6(x - 1) - 6(y - 2) + 3(z - 1) = 0$. \qed
Problem 4: (20 points) Let $F(x, y, z) = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$. Let $c$ be the oriented curve from $(0, 0, 0)$ to $(1, 1, 1)$ that lies on the intersection of the surface $y = x^2$ with the surface $z = x$. Evaluate the line integral $\int_c F \cdot ds$.

Solution. We can parameterize $c$ by $\mathbf{c}(t) = (t, t^2, t)$ for $0 \leq t \leq 1$. Then the line integral becomes

$$\int_c F \cdot ds = \int_0^1 F(\mathbf{c}(t)) \cdot \mathbf{c}'(t) dt$$

$$= \int_0^1 (t^3, t^2, -t^3) \cdot (1, 2t, 1) dt$$

$$= \int_0^1 2t^3 dt$$

$$= \frac{1}{2}.$$
Problem 5: (20 points)

Let \( \mathbf{F}(x,y,z) = x^2 \mathbf{j} - xz \mathbf{k} \). Let \( S \) be the surface consisting of all the points \((x, y, z)\) that satisfy \( y = x^2, -1 \leq x \leq 1, \) and \( 0 \leq z \leq 2 \).

Find the inward flux of \( \mathbf{F} \) across \( S \)—that is, pointing toward the \( yz \) plane.

Solution. We parameterize \( S \) by \( \Phi(u,v) = (u, u^2, v) \) for \(-1 \leq u \leq 1 \) and \( 0 \leq v \leq 2 \). Then \( T_u = (1, 2u, 0) \) and \( T_v = (0, 0, 1) \), so that \( T_u \times T_v = (2u, -1, 0) \). Note that this points \textit{outwards}, so we need to work with \(-T_u \times T_v = T_v \times T_u\). Then the flux integral is just

\[
\begin{align*}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^2 \int_{-1}^1 \mathbf{F}(\Phi(u,v)) \cdot (T_v \times T_u) \, du \, dv \\
&= \int_0^2 \int_{-1}^1 (0, u^2, uv) \cdot (-2u, 1, 0) \, du \, dv \\
&= \int_0^2 \int_{-1}^1 u^2 \, du \, dv \\
&= \frac{4}{3}
\end{align*}
\]

\( \square \)
Problem 6: (20 points)

Suppose we want to maximize a production function, $f(x, y) = x^{2/3} y^{1/3}$, under the budget constraint $x + y = 3.78$. Use Lagrange multipliers to find the values of $x$ and $y$ that maximize $f$. (You do not need to justify that the solution is a maximum.)

Solution. This is a Lagrange multipliers problem. We let $g(x, y) = x + y$ and we need to solve the equation $\nabla f = \lambda \nabla g$. That is, we need to solve

$$\frac{2}{3} \left( \frac{y}{x} \right)^{1/3} = \lambda, \quad \frac{1}{3} \left( \frac{x}{y} \right)^{2/3} = \lambda$$

simultaneously. Note that $\nabla f$ is not defined at $(0, 0)$, but this is no worry because $0 + 0 = 0 \neq 3.78$. In particular, this shows that none of $x$, $y$, or $\lambda$ are equal to zero. Thus, we have

$$2 \left( \frac{y}{x} \right)^{1/3} = \left( \frac{x}{y} \right)^{2/3}.$$

Multiplying both side of the equation by $\left( \frac{x}{y} \right)^{1/3}$, we see that $2 = \frac{x}{y}$ or $x = 2y$. Thus, the value of $(x, y)$ which maximizes $f$ is $(\frac{2 \times 3.78}{3}, \frac{3.78}{3})$. \qed
Problem 7: (20 points)

A solid region is bounded by the graph of \( z = 8 - 2x^2 - 2y^2 \) and the \( xy \) plane.

a) Sketch the region.

Solution. This is an upside-down paraboloid which intersects the \( z \)-axis at the point \((0, 0, 8)\) and intersects the \( xy \)-plane in the circle \( x^2 + y^2 = 4 \).

b) Find the volume of the region.

Solution. Since the region is the graph of the function \( g(x, y) = 8 - 2x^2 - 2y^2 \), the volume is just given by the double integral over the disk \( D : x^2 + y^2 \leq 4 \)

\[
\int_{D} \int (8 - 2x^2 - 2y^2) \, dx \, dy = \int_{0}^{2\pi} \int_{0}^{2} (8 - 2r^2) r \, dr \, d\theta
\]

\[
= 2\pi \int_{0}^{2} 8r - 2r^3 \, dr
\]

\[
= 2\pi \left( 4r^2 - \frac{1}{2}r^4 \right) \bigg|_{0}^{2}
\]

\[
= 16\pi.
\]

\( \blacksquare \)

c) Find the surface area of the region.

Solution. Since the surface is the graph of \( g(x, y) \), we can use \((-g_x, -g_y, 1)\) as our normal vector, so that the surface area is

\[
\int_{S} dS = \int_{D} \|(4x, 4y, 1)\| \, dx \, dy
\]

\[
= \int_{D} \sqrt{16(x^2 + y^2) + 1} \, dx \, dy
\]

\[
= \int_{0}^{2\pi} \int_{0}^{2} r \sqrt{16r^2 + 1} \, dr \, d\theta
\]

\[
= 2\pi \int_{0}^{2} r \sqrt{16r^2 + 1} \, dr
\]

\[
= \frac{\pi}{16} \left( 16r^2 + 1 \right)^{3/2} \bigg|_{0}^{2}
\]

\[
= \frac{3\pi}{8} (65^{3/2} - 1).
\]

\( \blacksquare \)
Finally, we have to add in the surface area coming in from the base of the figure, which is $4\pi$. Thus, we get that the total surface area is

$$\frac{3\pi}{8} (65^{3/2} - 1) + 4\pi.$$
Problem 8: (25 points—5 for each part)

\( \mathbf{F} = 3x \mathbf{i} + z \mathbf{j} + 5z \mathbf{k} \) and \( S \) is the surface of the half-circular cylinder shown below. The radius of the circle is 4 and the height of \( S \) is 5. Name the faces of the surface \( S \) as follows:

The top face, lying in the plane \( z = 5 \) is \( S_1 \); the bottom face, lying in the \( xy \) plane, is \( S_2 \); the vertical face in the \( (x, z) \) plane is \( S_3 \); and the curved vertical face is \( S_4 \).

\begin{align*}
\text{a) Evaluate the flux } & \iint_{S} \mathbf{F} \cdot \mathbf{n} \, dS \text{ outward through the entire surface } S. \\
\end{align*}

Solution. By the Divergence Theorem

\[
\iint_{S} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_{V} \nabla \cdot \mathbf{F} \, dV \\
eq \iiint_{V} 8 \, dV \\
eq 8 \text{(volume of } V) \\
eq 8 \cdot \frac{1}{2} \cdot 5 \cdot \pi \cdot (4^2) \\
eq 320\pi.
\]

\( \square \)
b) Evaluate the circulation (in the counterclockwise direction) around the boundary of $S_1$.

**Solution.** Note that to get a counterclockwise direction, we orient $S_1$ with the upward pointing normal vector. Also, $\nabla \times F = (-x, 0, z)$ so that by Stokes' Theorem, the circulation is

$$\int_{\partial S_1} F \, ds = \iint_{S_1} (\nabla \times F) \cdot \mathbf{n} \, dS$$

$$= \iint_{S_1} (-x, 0, z) \cdot (0, 0, 1) \, dS$$

$$= \iint_{S_1} z \, dS$$

$$= \iint_{S_1} 5 \, dS, \text{ since we are in the plane } z = 5$$

$$= 5 \text{(Area of } S_1)$$

$$= 40\pi.$$

\[\square\]

c) Find the flux of $\nabla \times F$ across the portion of $S$ that does not include $S_1$.

**Solution.** The flux of $\nabla \times F$ across $S - S_1$ is its flux across $S$ minus its flux across $S_1$. But since $S$ is a closed surface, the flux of $\nabla \times F$ across $S$ is zero. Thus, we have that the flux of $\nabla \times F$ across $S - S_1$ is negative its flux across $S_1$. By Stokes' Theorem and part (b), that value is $-40\pi.$

\[\square\]
d) Let $D_1$ be a subset of $S_1$ obtained by cutting out a circular hole. Specifically, let $c$ be a circle of radius 1 that lies wholly within $S_1$ (and doesn’t touch its boundary) and let $D_1$ be the result of cutting out the interior of the circular disk bounded by $c$. Give $D_1$ the same orientation as $S_1$ and orient its boundary consistently. Either find the circulation of $\mathbf{F}$ around the boundary of $D_1$ or explain why the information given is not enough to find it.

Solution. This problem is no different than part (b). We use Stokes’ Theorem to compute that the circulation is

$$
\int_{\partial D_1} \mathbf{F} \cdot d\mathbf{s} = \iint_{D_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS
$$

$$
= \iint_{D_1} (-x, 0, z) \cdot (0, 0, 1) \, dS
$$

$$
= \iint_{D_1} z \, dS
$$

$$
= \iint_{D_1} 5 \, dS, \text{ since we are in the plane } z = 5
$$

$$
= 5(\text{Area of } D_1)
$$

$$
= 5((\text{Area of } S_1) - (\text{Area of unit disk}))
$$

$$
= 40\pi - 5\pi
$$

$$
= 35\pi.
$$

\[ \square \]

e) Let $V$ be a 3-dimensional region whose outer surface is $S$ and whose inner surface is a sphere of radius 1 that lies entirely within the region enclosed by $S$ (and does not touch $S$). In part (a) you computed the flux of $\mathbf{F}$ outward across $S$. Either find the flux of $\mathbf{F}$ outward across the surface of $V$, or explain why the information given is not enough to find it.

Solution. From part (a), the outward flux through $S$ is $320\pi$. Using the Divergence Theorem, the inward flux through the unit sphere is

$$
-\iiint_{\text{unit sphere}} \nabla \cdot F \, dV = -\iiint_{V} 8 \, dV
$$

$$
= -8(\text{Volume of the unit sphere})
$$

$$
= -\frac{32\pi}{3}.
$$

Thus, the outward flux through the surface of $V$ is $320\pi - \frac{32\pi}{3}$.  \[ \square \]
Problem 9: (20 points)

Throughout this problem the temperature \( T \), measured in degrees C, in a metal ball of radius \( a \) meters is proportional to the square of the distance from the center of the ball. Taking the center of the ball as the origin, this means \( T(x, y, z) = C(x^2 + y^2 + z^2) \) where \( C \) is the proportionality constant.

a) Find the average temperature throughout the ball. The average temperature is given by

\[
\frac{\iiint_B T(x, y, z)\,dV}{\iiint_B dV}.
\]

The denominator is just the volume of the ball which is \( \frac{4a^3\pi}{3} \). We compute the top integral using spherical coordinates:

\[
\iiint_B T(x, y, z)\,dV = C \iiint_B x^2 + y^2 + z^2\,dV
\]

\[
= C \int_0^{2\pi} \int_0^\pi \int_0^a \rho^4 \sin \phi d\rho d\phi d\theta
\]

\[
= 2C\pi \int_0^\pi \sin \phi d\phi \int_0^a \rho^4 d\rho
\]

\[
= 2C\pi \cdot 2 \cdot \frac{a^5}{5}.
\]

Thus, the average temperature is

\[
\frac{\frac{4Ca^5\pi}{5}}{\frac{4a^3\pi}{3}} = \frac{3Ca^2}{5}.
\]

b) Find the average temperature on the surface of the ball.

Solution. The temperature on the surface of the ball is constant. Thus, its average value is the same as its value everywhere, \( Ca^2 \).

\( \Box \)

c) The point \((1,2,4)\) is in the interior of the ball. At \((1,2,4)\) in which direction does the temperature increase the fastest?

Solution. The gradient gives the direction of fastest increase. Since \( T(x, y, z) \) is proportional to the distance from \((x, y, z)\) to the origin, the gradient points radially outward. Thus, the direction of fastest increase at \((1,2,4)\) is \((1,2,4)\).

\( \Box \)
d) Use a linear approximation to estimate the change in temperature at (1,2,4) if you increase the x coordinate by .5, decrease the y coordinate by .3 and increase the z coordinate by .1.

Proof. The linear approximation is given by

\[ L(x, y, z) = T(x, y, z) + \nabla T(1, 2, 4) \cdot (x - 1, y - 2, z - 4). \]

In this case, this gives

\[
\begin{align*}
L(1, 2, 4) &= C(1 + 4 + 16) + C(2, 4, 8) \cdot (.5, -.3, .1) \\
&= 21C + .6C \\
&= 21.6C.
\end{align*}
\]

Thus, the change is .6C degrees. \( \square \)
Problem 10: (20 points)

a) In the diagram below, $S$ is the 2-dimensional region consisting of the area surrounded by curve $c_1$, but with the point $x$ removed. Each of the curves $c_1, c_2, c_3$ is oriented in the direction indicated by the arrowhead attached to it.

\[
\begin{align*}
F : S &\subset \mathbb{R}^2 \to \mathbb{R}^2 \text{ is a } C^2 \text{ function about which you’re given two pieces of information: } \\
\int_{c_1} F \cdot ds &= 3 \text{ and, on } S, \nabla \times F = 0. \text{ For each of the following two line integrals, either calculate its value (showing your work) or explain briefly why the given information is not sufficient to calculate it.} \quad a1) \int_{c_2} F \cdot ds
\end{align*}
\]

Solution. By Stokes’ Theorem, we know that

\[
\int_{c_1} F \cdot ds - \int_{c_2} F \cdot ds = \iint_{S-\text{Area inside } c_2} (\nabla \times F) \cdot n dS.
\]

This last integral is zero by assumption, so we must have that

\[
\int_{c_2} F \cdot ds = 3.
\]
a2) $\int_{c_3} \mathbf{F} \cdot d\mathbf{s}$

Solution. Again using Stokes’ Theorem, we know that

$$\int_{c_3} \mathbf{F} \cdot d\mathbf{s} = \iint_{\text{Area enclosed by } c_3} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$ 

Again, the second integral is zero, thus the first is too. \hfill \Box

b) In the diagram below, $V$ is the 3-dimensional region that lies between the two concentric cylinders and is bounded top and bottom by horizontal planes. $S$ is the surface of $V$, oriented outward.

Let $f : V \subset \mathbb{R}^3 \to \mathbb{R}^1$ be a $C^2$ function defined on $V$, and let $\mathbf{F} = \nabla f : V \subset \mathbb{R}^3 \to \mathbb{R}^3$ be the gradient of $f$. Indicate whether the following two statements are true or false, with a brief explanation of your answer:

b1) For every simple closed curve $c$ lying wholly in $V$, $\int_c \mathbf{F} \cdot d\mathbf{s} = 0$.

Solution. True: $\mathbf{F}$ is a gradient field, and gradient fields are conservative. \hfill \Box

b2) The divergence theorem says that the outward flux of $\mathbf{F}$ through $S$ is $\iint_V \text{div} \mathbf{F}$, and therefore implies that the flux is zero.

Solution. False: The Divergence Theorem says that the outward flux of $\mathbf{F}$ through $S$ is $\iiint_V \text{div} \mathbf{F}$. However, $\nabla \times \mathbf{F} = 0$ gives no reason to believe that $\nabla \cdot \mathbf{F}$ or the integral of $\nabla \cdot \mathbf{F}$ equals 0. Consider, for example $\mathbf{F}(x, y, z) = (x, y, z)$. \hfill \Box