Convergence of some time inhomogeneous Markov chains via spectral techniques

Jessica Zuniga
with Laurent Saloff-Coste

Motivation

- Mironov (2002)
  (Not so) random shuffles of RC4.
  - *total variation upper bound of $O(n \log n)$ for cyclic-to-random shuffle*
  - *lower bound of $O(n)$ for cyclic-to-random shuffle*
- *Mossel, Peres, Sinclair (2004)*
  Shuffling by semi-random transpositions.
  - total variation upper bound of $O(n \log n)$ for any semi-random transposition shuffle
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$$P(X_n = x | X_{n-1} = y, X_{n-2} = x_{n-2}, \ldots, X_0 = x_0)$$
$$= P(X_n = x | X_{n-1} = y)$$
$$= K_n(x,y)$$

Let μ_0 be the distribution of X_0. The distribution μ_n of X_n is

$$\mu_n(x) = \sum_{x \in V} \mu_0(x) K_{0,n}(x,y)$$

where $K_{n,n} = I$ and for $m \geq n$

$$K_{n,m}(x,y) = \sum_{z \in V} K_{n,m-1}(x,z) K_m(z,y)$$

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Major assumption: There exists a measure π that is invariant for the sequence $(K_n)_1^\infty$, i.e.

$$\pi K_n = \pi.$$

Large class of examples: random walks on groups.

Let $V = G$ a finite group equipped with a probability measure μ. The Markov kernel

$$K(x, y) = \mu(x^{-1}y)$$

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Definition: Let $\mathcal{Q} = \{Q_1, \ldots .Q_n\}$ have invariant measure π. $(\mathcal{Q}, \pi)$ is ergodic if for any $(K_i)_1^\infty$ with invariant measure π and $K_i \in \mathcal{Q}$ for infinitely many i then

$$\lim_{n\to\infty} K_{0,n}(x,z) - K_{0,n}(y,z) = 0$$

for all $x, y, z \in V$.

Remark:

- Let $\mathcal{Q} = \{Q\}$. $(\mathcal{Q}, \pi)$ is ergodic is stronger than

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Distances on $\ell^2(V, \pi)$:

$$d_{TV}(\mu, \pi) = \sup_{A \subset V} |\mu(A) - \pi(A)|$$

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$$\sigma_i(K) = \sqrt{\beta_i(KK^*)} = \sqrt{\beta_i(K^*K)}.$$

Theorem: Consider (K, π). Let $(\psi_i)_0^{|V|-1}$ be a basis of $\ell^2(V, \pi)$ such that $KK^*\psi_i = \sigma_i(K)^2\psi_i$ (assume $\psi_0 = 1$.) Then

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Main technical result

Theorem Let $(K_i)_1^\infty$ be a sequence in V with positive invariant measure π. For $j \geq 1$ and $0 \leq i \leq |V| - 1$ let $\sigma_i(K_j)$ be the singular values of K_j on $\ell^2(V, \pi)$ then

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For all $k = 1, \ldots, |V| - 1$

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  (1) $\sigma_1(Q) < 1$ implies $Q^n(x, \cdot) \to \pi$ for all x

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Theorem Let $\mathcal{Q} = \{Q_1, \ldots, Q_k\}$ be a family of kernels on V with invariant measure π.

- $(\mathcal{Q}, \pi)$ is ergodic iff $\sigma_1(Q_j) < 1$ for all $1 \leq j \leq k$.

- If $(\mathcal{Q}, \pi)$ is ergodic then for any $(K_i)_1^\infty$ with invarient measure π and infinitely many $K_i \in \mathcal{Q}$ we have

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Application to time inhomogeneous random walks on finite groups

Main technical result simplifies when $V = G$, in this case $\pi = 1/|G|$.

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- For $0 \le j \le |G| - 1$ set $\sigma_j(p_i) = \sigma_j(K_i)$ where singular values are taken w.r.t. π

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Application to time inhomogeneous random walks on finite groups

Main technical result simplifies when $V = G$, in this case $\pi = 1/|G|$.

- $(p_i)_1^\infty$ a sequence of probability measures on G.
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Example: $(S_j)_1^\infty$ a sequence of generating sets of G s.t. $\mathrm{id} \in S_j$. Set

$$K_j(x,y) = \begin{cases} 1/|S_j| & \text{if } x^{-1}y \in S_j \\ 0 & \text{otherwise.} \end{cases}$$

Well know eigenvalue estimates give us

$$\sigma_1(K_j)^2 \le 1 - \frac{1}{|S_j|^2 d_j^2}$$

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Information can be lost with the inequality

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$c_{i,j}$: pick card in position i and insert it into position j.

$$c_{i,j} = \begin{cases} \mathrm{id} & \text{if } i = j \\ (j, j-1, \ldots, i+1, i) & \text{if } 1 \leq i < j \leq n \\ (j, j+1, \ldots, i-1, i) & \text{if } 1 \leq j < i \leq n. \end{cases}$$

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Theorem For any $r = (r_i)_1^\infty$, $r_i \in \{1, \ldots, n\}$

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For $n > 28$, $c > 2$ and $k \geq 4n(\log n + c)$

$$d_2(\tilde{p}^r_{0,k}, \pi) \leq 2e^{-(c-2)}.$$

Proof

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