A Recursion-theoretic View of Axiomatizable Theories

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Modern mathematical practice is deeply committed to a study of axiom systems and their consequences. Accordingly an abstract, recursion-theoretic analysis of axiomatizable formal theories is of interest in a study of the foundations of mathematics. In this paper we will be concerned with:

A. A classification of mathematically interesting formal theories by deduction-preserving recursive isomorphisms (§ 2).

B. Effective extensibility, an abstraction of the Gödel incompleteness construction, and its relation to effective inseparability (§ 3).

C. The priority method tailored to formal theories (§ 4).

Our discussion will include results and especially open problems connected with these topics.

We begin our discussion with a brief historical sketch (§ 1). Unfortunately limitations of space prevent us from giving a more detailed summary: only those results which are intimately related to the considerations of this paper will be highlighted.

Unless otherwise stated (e.g. § 3) we assume that all theories are consistent, axiomatizable theories formulated as applied predicate calculi.

§ 1 Preliminary Considerations — Historical Sketch of Recursion Theory and Axiomatizable Theories.

It is well known that the early development of recursion theory was related to the study of formal theories by means of the well-known
incompleteness and undecidability results of Gödel [4] and Church [1a]. In order to explore some of these concepts abstractly Post [13] began to study recursively enumerable (r.e.) sets of natural numbers. From the point of view of formal theories this was a reasonable approach. For by Gödelization the set of theorems of an axiomatizable theory could be identified with an r.e. set of natural numbers. In an attempt to abstract the Gödel construction, Post defined a creative set 2. Next he showed that the creative set possesses the highest r.e. degree of unsolvability. Finally he concluded with the following open problem (Post's Problem): is there an r.e. set of natural numbers whose degree of unsolvability is neither the same as a recursive set nor a creative set?

In the years that followed attempts were made to classify r.e. sets. The hope was that this classification would help in the solution of Post's Problem. The properties of simple, hypersimple, mesoic etc. sets were extensively explored and discussed. Finally in 1956 Friedberg [3] and Muchnik [9] gave an affirmative solution to Post's Problem. It is interesting to note that the work of Friedberg and Muchnik did not depend upon the classification of r.e. sets referred to above. As is well-known they both employed an ingenious "priority argument". One consequence of the work of Friedberg and Muchnik was that much of the activity concerning the classification of r.e. sets ceased and was replaced by an energetic exploration of the intricacies of the priority method. The vast bulk of this work has had little relation to formal theories.

In an attempt to relate Post's Problem to formal theories Feferman [2] and Shoenfield [15] exhibited undecidable theories which did not have the highest r.e. degree of unsolvability. In contrast it follows from the work of Feferman [2] and Smullyan [16] that many of the mathematically interesting theories—e.g. set theory and first order number theory—are not only creative but also effectively inseparable—and hence of the highest degree of unsolvability. Thus it becomes natural to consider from the recursion-theoretic point of view the problem of a meaningful classification of creative and effectively inseparable theories (§ 2). For reasons which will become apparent in § 2 we consider creative theories only briefly: our attention will eventually focus on effectively inseparable theories.
§ 2 Classification of Effectively Inseparable Theories by Deduction-preserving Recursive Isomorphisms.

By 1955 the implicit interest in classifying formal theories by recursion-theoretic considerations had given rise to a more explicit formulation. In that year Myhill [10] showed that any two creative sets are recursively isomorphic\(^5\). On the basis of this result Myhill referred to any two creative theories as "notational variants" of each other. Thus according to Myhill any two consistent extensions of the theory \(\mathcal{R}\) of Undecidable Theories \([17]\)—e.g. set theory and first order number theory—are "notational variants".

Very shortly afterwards the following stronger result appeared \([16]\): any two pairs of effectively inseparable sets are "doubly recursively isomorphic"\(^6\). Thus by a strengthened version of Myhill's criterion—viz! that double recursive isomorphism yields "notational variants"—any two effectively inseparable theories would be "notational variants" of each other. Since \(\mathcal{R}\) and all its extensions\(^7\) are not merely creative theories but also effectively inseparable, they would again be considered "notational variants".

The author would like to offer another point of view \([cf.\,1]\). The results of Myhill and Smullyan seem to suggest, not that any two effectively inseparable theories are notational variants of each other, but rather that the concept of double recursive isomorphism—and hence of recursive isomorphism—is too crude to provide a meaningful classification for formal theories. One suggestion is that the concept of (double) recursive isomorphism is not adequate because it fails to take into account the deductive structure of the formal theories involved. Accordingly it seems natural to consider deduction-preserving recursive isomorphisms. Recent results of Kripke and the author \([12]\) indicate that these isomorphisms give evidence of the beginning of a classification of effectively inseparable theories in standard formalization. This evidence is obtained as a consequence of the following results.

Let \(J_1\) and \(J_2\) be two theories in standard formalization. Let \(W_i\) be the set of sentences of \(J_i\) \((i = 1, 2)\). By a D.I.N.\(^8\) we mean a \(1 \rightarrow 1\) function mapping \(W_1\) onto \(W_2\) preserving deducibility, implication and negation. (Note that a D.I.N. also preserves theoremhood and undecidability). Then

1. If \(J_1\) and \(J_2\) are effectively inseparable there is a recursive D.I.N. mapping \(W_1\) onto \(W_2\).
(2) If \( J_1 \) and \( J_2 \) contain a small fragment of arithmetic there is a primitive recursive D.I.N. mapping \( W_1 \) onto \( W_2 \).

In general we cannot replace "recursive" by "primitive recursive" in (1). This is a consequence of:

(3) Let \( F \) be an r.e. class of recursive functions. Then there exists an effectively inseparable theory \( J \) in standard formalization such that no recursive function which witnesses the effective inseparability of \( J \) is in \( F \).

By systematically exploiting (1) (2) and (3) we are able to classify all effectively inseparable theories in standard formalization into \( \aleph \) equivalence classes with a unique maximum element. Note that the mappings in (1) and (2) preserve modus ponens. Note also that these mappings are such that tautologies are mapped onto tautologies. As a consequence the mappings of (1) and (2) come very close to preserving the proof structure—i.e. to sending each proof of a theorem in \( J_1 \) into a corresponding proof of the corresponding theorem of \( J_2 \). This is especially clear if we assume that the theories are formulated so that modus ponens is the sole rule of inference [14]. Thus we are led to the following open problem.

**Prob. 1.** To what extent can the proof structure be completely preserved?

Unfortunately \( \aleph \) and all its consistent extensions belong to the unique maximum element. Hence our notion of deduction-preserving "recursive-isomorphism" is still too crude to provide an adequate classification for these theories. This leads to several open problems. We mention only a few. The reader can easily supply others.

**Prob. 2.** Can results (1) and (2) be extended so that formulas are mapped onto formulas such that if \( A \) is mapped onto \( B \), the closure of \( A \) is mapped onto the closure of \( B \) ?

**Prob. 3.** Can results (1) and (2) be extended so that formulas are mapped onto formulas in such a way that if \( J_1 \) and \( J_2 \) both possess numerals and \( F_1 (x) \) is mapped into \( F_2 (x) \), then \( F_1 (n) \) is mapped into \( F_2 (n) \) ?

**Prob. 4.** Can results (1) and (2) be extended so that axioms of the predicate calculus are mapped onto axioms of the predicate calculus?

Note that the above problems point in the direction of requiring that additional deductive structure be preserved. Hence these problems are related to problem 1. For this reason an affirmative answer to any
or all of problems 2-4 would be very interesting and somewhat surprising. A negative answer would be equally interesting—particularly if it could be used as the basis for a meaningful abstract classification of all consistent extensions of $\mathcal{R}$.

In concluding this section we wish to state one more open problem.

\textbf{Prob. 5.} Investigate the existence of "minimal subclasses" $S$ of the primitive recursive functions such that (2) holds with the mapping chosen from $S$.

It may very well be that suitable subclasses $S'$ of $S$ may be useful in obtaining a classification.

\section*{§ 3 Effectively extensible theories and the Gödel construction [11].}

It is well known that Gödel's famous undecidability result may be viewed in the following strong form. Suppose we are given a specific presentation (i.e. a specific formulation in terms of axioms and rules of inference) of number theory. Then there exists an effective method which, when applied to a consistent axiomatizable extension of the theory yields an undecidable sentence of this extension. For distinct presentations the undecidable sentences obtained would be distinct. This is because the sentence constructed depends upon the notion of proof and hence ultimately upon the axioms and rules of inference—i.e. upon the specific presentation.

The concept of effective extensibility (Pour-El [11]) is an attempt to provide an abstract formulation for this strong form of Gödel's theorem. We focus our attention, not on the concept of a theory, but on the concept of a \textit{presentation} of a theory, a concept which gives some insight into the theory as a deductive structure. By a \textit{theory} $\mathcal{J}$ we will mean an ordered quadruple $\langle F, W, T, \neg \rangle$ where $F$ and $W$ are recursive sets such that $W \subseteq F$, $T$ is a recursively enumerable set such that $T \subseteq W$ and $\neg$ is a strictly increasing recursive function mapping $F$ into $F$ and $W$ into $W$ such that

$$\neg \varphi \geq \varphi$$

$$\varphi \in T \iff \neg (\neg \varphi) \in T$$

Intuitively $F$ represents the (Gödel numbers of) the set of formulas, $W$ the set of sentences, $T$ the set of theorems and $\neg$ the negation operator. If we define $R$ by

$$R = \{ \varphi \mid \neg \varphi \in T \}$$

then it is easy to see that

$$\neg \varphi \in R \iff \varphi \in T$$
Hence $R$ represents the set of refutable sentences and $W - (T \cup R)$ represents the set of undecidable sentences. Thus all our theories are axiomatizable theories possessing a classical negation operator. Furthermore they are all assumed to be consistent.

As stated above we focus our attention, not on the concept of a theory, but on the concept of a presentation of a theory. Roughly, a presentation is an ordered pair $(a, R)$, $a$ a recursively enumerable set (the set of axioms), $R$ a recursively enumerable sequence of recursively enumerable relations (the rules of inference). Effective extensibility is defined in terms of the concept of a presentation. Roughly, a consistent theory is effectively extensible if there exists a presentation $(a, R)$ and a recursive function $f$ associated with this presentation such that if $\omega_1$ is a recursively enumerable set of sentences which can be added consistently to the theory, $f(i)$ is an undecidable sentence of the extension $(a \cup \omega_1, R)$.

How does effective extensibility compare with effective inseparability? Effective inseparability provides an invariant characterization for effective extensibility. More precisely:

1. If a theory is effectively extensible with respect to one presentation it is effectively extensible with respect to all.
2. A theory is effectively extensible if and only if it is effectively inseparable.

Thus effective inseparability and effective extensibility are equivalent. Whether or not further refinements of the concept of effective extensibility will lead to a meaningful abstract classification of mathematically interesting formal theories is an open question.

§ 4 The priority method applied to formal theories

No discussion of the relation between recursion theory and formal systems is complete without a brief glance at the priority method. Recently D. A. Martin and the author have proved the following (in *Journal of Symbolic Logic*, vol. 35, No 2, 1970).

1. There is an axiomatizable, undecidable theory $J$ in standard formalization all of whose axiomatizable extensions are finite extensions.
2. There is an axiomatizable, undecidable theory $J$ in standard formalization such that
   a. $J$ has a complete consistent decidable extension $J'$
b. Every axiomatizable extension \( J^+ \) of \( J \) is such that either
   (i) \( J^+ \) is a finite extension of \( J \)
   or
   (ii) \( J' \) is a finite extension of \( J^+ \)

The proof of (6) uses a "priority" argument applied to a recursive enumeration of all axiomatizable theories—not necessarily consistent—formulated in the language of \( J \). As expected deducibility considerations play a key role. In contrast the proof of (7) depends upon the existence of a maximal set: the only priority used is implicit in the construction of this set. Attempts to obtain a proof of (6) by the method of (7)—i.e. by confining priority techniques to r.e. sets—have failed. Hence the following open problem may be of some interest.

**Prob. 6.** Investigate the extension of the priority technique to formal theories.

**Remark:** There exist other aspects of formal theories which can be characterized by recursion-theoretic methods and results. For example: hypersimplicity provides a necessary and sufficient condition for non-independent axiomatization\(^{12}\). As a consequence we can give a general method for constructing axiomatizable theories which are not independently axiomatizable. To what extent other aspects of formal theories can be characterized by recursion-theoretic concepts remains open. In this connection it is interesting to recall that the invention of the hypersimple set had nothing to do with independent axiomatization: it arose as a response to a purely recursion-theoretic phenomenon—reducibility considerations (see [13]). Although this paper was received by the editors in 1967, and must therefore be considered in the light of knowledge at that time, the open problems stated herein appear to be open as of this date.

**APPENDIX**

Let \( \omega_0, \omega_1, ..., \omega_n, ... \) be a fixed standard recursive enumeration of all recursively enumerable sets [6]. Let \( j, K, L \) be the usual pairing function and its inverses.

**Definition 1.** A recursively enumerable set \( a \) is creative if there exists a recursive function \( f \) such that if \( \omega_i \subseteq a \), then \( f(i) \notin a \cup \omega_i \).

**Definition 2.** A theory is creative if the set (of Gödel numbers) of its theorems is creative.
Definition 3. A pair of disjoint recursively enumerable sets \((a, \beta)\) is **effectively inseparable** if there exists a recursive function \(f\) such that if
\[
\omega_{K(i)} \supseteq a, \quad \omega_{L(i)} \supseteq \beta, \quad \omega_{K(i)} \cap \omega_{L(i)} = \emptyset
\]
then
\[
f(i) = \omega_{K(i)} \cup \omega_{L(i)}
\]

Definition 4. A theory is **effectively inseparable** if the pair \((T, R)\) of theorems and refutables form an effectively inseparable pair of sets.

Definition 5. A presentation of a theory \(J\) is an ordered pair \((a, R)\), \(a\), a recursively enumerable subset (the set of axioms) of \(T, R\), a recursively enumerable sequence \(S_0, S_1, \ldots\) of recursively enumerable relations each of rank \(\geq 2\) (the rules of inference) such that if \(S_n\) is a \(k\)-place relation, \(S_n \subseteq F^k\). Furthermore if and only if \(\varphi \in T\) if and only if \(\varphi \in W\) and there exists a finite sequence \(\varphi_1, \ldots, \varphi_n\) of elements of \(F\) such that \(\varphi_n = \varphi\) and for each \(i\), either \(\varphi_i \in a\) or there exists an \(S_n\) of rank \(p_n + 1\) and a subsequence \(\varphi_{i_1}, \ldots, \varphi_{i_k}\) \((i_k < i\) for \(k = 1, \ldots, p_n\)) such that \(S_n(\varphi_{i_1}, \varphi_{i_2}, \ldots, \varphi_{i_k})\).

**Notation**
The concept of a presentation gives rise to the following notation.
- \(\text{Cl}(a, R)(\omega_i) = \) the set of all theorems of the theory with axioms \(a \cup \omega_i\) and with rules of inference \(R\) (the closure of \(a \cup \omega_i\) under \(R\)).
- \(\text{R}(a, R)(\omega_i) = \) the set of refutable sentences of the theory with axioms \(a \cup \omega_i\) and with rules of inference \(R\).

We will need the following definitions:

**Definition 6:** a) A theory \(J\) is **consistent** if \(T \cap R = \emptyset\). Thus for every presentation \((a, R)\) of \(J\), \(\text{Cl}(a, R)(\emptyset) \cap \text{R}(a, R)(\emptyset) = \emptyset\).
b) Let \(\omega_i \subseteq W\). A theory \(J\) is **consistent with** \(\omega_i\) with respect to presentation \((a, R)\) if \(\text{Cl}(a, R)(\omega_i) \cap \text{R}(a, R)(\omega_i) = \emptyset\).

**Definition 7:** A theory \(J\) is **essentially consistent with respect to presentation** \((a, R)\) whenever the following holds: let \(\omega_i \subseteq W\); if \(J\) is consistent with \(\omega_i\) with respect to \((a, R)\) and if \(\varphi \in W - \text{R}(a, R)(\omega_i)\), then \(J\) is consistent with \(\{\varphi\}\) with respect to \((a \cup \omega_i, R)\).

**Definition 8:** A presentation \((a, R)\) of a consistent theory \(J\) is **effectively extensible** (e.e.) if there is a recursive function \(f\) such that whenever \(\omega_i \subseteq W\) and
\[
\text{Cl}(a, R)(\omega_i) \cap \text{R}(a, R)(\omega_i) = \emptyset
\]
then
\[
f(i) \in W - \{\text{Cl}(a, R)(\omega_i) \cap \text{R}(a, R)(\omega_i)\}
\]

**Definition 9:** A consistent theory \(J\) is **effectively extensible** if it has an effectively extensible presentation.
FOOTNOTES

1. Partially supported by NSF GP-5434.
2. A list of some of the basic definitions can be found in the appendix.
3. Post seems to have regarded the creative set as a recursion-theoretic abstraction of the Gödel incompleteness procedure.
4. Effectively inseparable pairs of sets also represent a recursion-theoretic abstraction of the Gödel incompleteness theorem. For a comparison between creativity and effective inseparability see definitions 1-4 of the appendix. Recall that all effectively inseparable theories are creative but not conversely.
5. That is - if \( a_1 \) and \( a_2 \) are creative then there is a 1-1 recursive function mapping the set of natural numbers onto itself and mapping \( a_1 \) onto \( a_2 \).
6. That is - if \( (a_1, \beta_1) \) and \( (a_2, \beta_2) \) are two effectively inseparable pairs of sets there is a 1-1 recursive function mapping the set of natural numbers onto itself and mapping \( a_1 \) onto \( a_2 \), \( \beta_1 \) onto \( \beta_2 \).
7. See Undecidable Theories [17] for notation.
8. D.I.N. is an abbreviation for deducibility-implication-negation-preserving.
9. More precisely - \( J_1 \) and \( J_2 \) contain two subtheories \( J'_1 \) and \( J'_2 \), a notation for numerals, a binary predicate \( \leq \) such that
   1) \( \vdash (x \leq \bar{a} \rightarrow .x = O .v . .v . x = \bar{a}) \)
   2) \( \vdash (x \leq \bar{a} .v . \bar{a} \leq x) \)
   3) Every primitive recursive function of one argument is definable in \( J'_1 \).
   Note that any theory containing this fragment of arithmetic is, a fortiori, effectively inseparable.
9 a. A similar classification holds for effectively inseparable theories formulated merely as applied predicate calculi.
10. For precise definitions see the appendix.
11. In this paper we consider only presentations of a theory \( J \) such that \( J \) is essentially consistent with respect to these presentations (see definition 7 of the appendix).
12. More precisely, let \( J \) be an axiomatizable theory of the applied predicate calculus. Then the following are equivalent
   i) \( J \) is not independently axiomatizable.
ii) Given any recursively enumerable axiomatization \( a_0, a_1, \ldots \) for \( J \) the set \( A \) defined by
\[
A = \{ i \mid a_i \text{ is deducible in the predicate calculus from the } a_j \text{'s for } j < i \} \text{ is hypersimple}
\]
iii) There exists an axiomatization \( a_0, a_1, \ldots a_n, \ldots \) for \( J \) such that the set \( A \) defined in ii) is hypersimple.

BIBLIOGRAPHY


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