HOMEWORK SOLUTIONS MATH 432 ASSIGNMENT 6

Exercise 2.98

Let $Z(G) = H$ and suppose G/H is generated by aH. Then for any element $b \in G$, it is in some of $(aH)^r$ therefore could be written as $a^r h$ for $h \in H$. Now $ab = aa^r h = a^r ha = ba$ because H is the center. That means a is also in the center. That could only happen if G/H is trivial (contains only one element), i.e. G is abelian.

Exercise 2.100

Let $X = \{x_1, x_2, ..., x_n\}$ and let $H = \langle x_n \rangle$. We see G/H is generated by $n-1$ elements. Let $\pi : G \to G/H$ be the projection map, then $\pi(S)$ will be a subgroup of G/H, so by induction it is finitely generated by X'. Now let y generates $S \cap H$, and for each $x \in X'$, let x^* be an element in S whose image is x. Finally check that y and those x^*'s generate S.

Exercise 2.107

First notice that $aK \cap Ka$ is nonempty. So by Ex 2.53(ii) $Kb = Ka$ as in the assumption. Therefore $aK = Ka$, K is normal.

Exercise 2.108

$$ab = \mu(a, b) = \mu(e, b)\mu(a, e) = ba$$

The first the the third equality comes from definition, and the second one is from homomorphism.

Exercise 2.113

(i) Notice for any $g \in G$, $h \in G'$, we see $gh = gh(g^{-1}h^{-1}hg) = (ghg^{-1}h^{-1})hg$, therefore left ocset gG' equals the right one $G'g$.

(ii) For any $g_1, g_2 \in G$, we see $g_1 g_2 = (g_1 g_2 g_1^{-1} g_2^{-1}) g_2 g_1$, so $g_1 g_2$ is in the coset of $g_2 g_1$, that means G/G' is abelian.

(iii) Suppose G' is not a subgroup of $im\varphi$, then there exist x, y s.t. $\varphi(xyx^{-1}y^{-1}) \neq e$. Therefore we get $\varphi(x)\varphi(y)(\varphi(x))^{-1}(\varphi(y))^{-1} \neq e$, or $\varphi(x)\varphi(y) \neq \varphi(y)\varphi(x)$, contradiction ($A$ is abelian).

Suppose $im\varphi$ is not abelian, then there exist x, y s.t. $\varphi(x)\varphi(y) \neq \varphi(y)\varphi(x)$, easy to check that $xyx^{-1}y^{-1}$ is not in the kernel, contradiction.

(iv) Almost the same as the proof of (i).