HOMEWORK SOLUTIONS MATH 432 ASSIGNMENT 2

Exercise 2.23

Suppose σ has t cycles, we see $sgn(\sigma) = (-1)^{n-t}$. Now σ' has $t-1$ cycles by removing the 1-cycle of j from σ, so $sgn(\sigma') = (-1)^{n-1-(t-1)} = sgn(\sigma)$.

Exercise 2.27

It is not difficult to verify:

$$(i,j) = (i,i+1)(i+1,i+2)...(j-2,j-1)(j,j-1)(j-1,j-2)...(i+2,i+1)(i+1,i)$$

Or you can use induction on $|i-j|$, and use a similar fact:

$$(i,j+1) = (i,j)(j,j+1)(i,j)$$

Exercise 2.32

Consider a map $f : O_n \to A_n$ from the odd permutations to the even ones that maps each σ to $\tau\sigma$ where τ is any transposition. By Lemma 2.38 this map is well-defined so it is sufficient to show that it is both injective and surjective.

Injective: suppose σ and σ' are mapped to the same element η, i.e.

$$\tau\sigma = \eta = \tau\sigma'$$

Now multiplying $\tau^{-1} = \tau$ to both sides, we get $\sigma = \sigma'$. This means f is injective.

Surjective: easy to see for any $\eta \in A_n$, $\tau^{-1}\eta$ is an element in O_n whose image is η.

So finally f is bijective, therefore there are same number of odd permutations as even ones, both counts as $\frac{1}{2}n!$. (Note you can also use Prop 2.10 to prove bijection.)

Exercise 2.34

Suppose $\alpha \neq id$ commutes with every element in S_n. Because it is not identity, we have $\alpha(i) = j$ for $i \neq j$, i.e. it maps i to a different j. Now $n \geq 3$, we have at least a third element, namely k, which is different from both i and j. Now consider a transposition $\beta = (j,k)$. By our assumption we should have $\alpha\beta = \beta\alpha$, but it is easy to check that $\alpha\beta$ maps i to j and $\beta\alpha$ maps i to k, contradiction.

Exercise 2.39(i)

First of all, if a permutation has order 2, then it consists of at least one 2-cycles and some number of 1-cycles.

So for $n = 5$, there are two possibilities: one 2-cycle or two 2-cycles. In the first case, we are choosing 2 element from 5, therefore the number of possibilities is $\binom{5}{2}$. In the second case, we need two 2-cycles, i.e. two groups of two elements. First we choose 2 from 5, that is $\binom{5}{2}$, next step we choose 2 from 3 elements left, that is $\binom{3}{2}$. Finally notice that you can switch the order of the two groups chosen and get the same permutation, so you need divide that by 2!. The final answer is:

$$\binom{5}{2} + \frac{\binom{5}{2}\binom{3}{2}}{2!} = 25$$

Use the similar argument for $n = 6$, only notice that there are 3 possibilities. The answer is:

$$\binom{6}{2} + \frac{\binom{6}{2}\binom{4}{2}}{2!} + \frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!} = 75$$

Exercise 2.40

First of all $(y^t)^d = y^{td} = y^m = e$, so the order of that is at most d. Now for any $x < d$, if $(y^t)^x = y^{tx} = e$, that will contradict the fact that y has order $m = dt$. So finally y^t has order d.

Exercise 2.44

Suppose not, that means there are two elements x and y s.t. $xy \neq xy$, then by multiplying xy to both sides, we get $(xy)(xy) = xyxy \neq yxxy = y(xx)y = yy = e$, contradiction.

Exercise 2.48

Matrix Product:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} e & g \\ f & h \end{pmatrix} = \begin{pmatrix} ae + cf & ag + ch \\ be + df & bg + dh \end{pmatrix}$$

Now easy to verify that $ae + cf + be + df = 1$ and $ag + ch + bg + dh = 1$.

Inverse:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

Just notice $d - b = d(a + b) - b(c + d) = ad - bc$, similarly $a - c = ad - bc$.