4. (4 points) Solution:
(a) 
\[ \int_C \frac{2dz}{z^2 + 4iz - 1} = \int_0^{2\pi} \frac{2ie^{i\theta} d\theta}{e^{2i\theta} + 4ie^{i\theta} - 1} = \int_0^{2\pi} \frac{d\theta}{2 + \sin \theta} \]

\( z^2 + 4iz - 1 \) has two roots \(-2 + \sqrt{3}i\) and \(-2 - \sqrt{3}i\). Only \((\sqrt{3} - 2)i\) is inside the unit disc. By (2.4), \( Resf((\sqrt{3} - 2)i) = 1/\sqrt{3} \). So \( \int_0^{2\pi} \frac{d\theta}{2 + \sin \theta} = 2\pi/\sqrt{3} \).

(b) 
\[ \int_0^{2\pi} \frac{d\theta}{2 + \sin \theta} = \int_0^{2\pi} \frac{d\theta}{2 + \cos(\theta - \pi/2)} = \int_0^{3\pi/2} \frac{d\alpha}{2 + \cos \alpha} + \int_{-\pi/2}^{0} \frac{d\alpha}{2 + \cos(\alpha + 2\pi)} = \int_0^{2\pi} \frac{d\alpha}{2 + \cos \alpha} = 2\pi/\sqrt{3} \]

where the last step is by (2.7).

6. (4 points) Proof: Just follow the hint. First of all, LHS of the equality in the hint is \( 2\pi i \times Res(f; i) = 2\pi i \times 1/(2i) = \pi \). The first term of RHS tends to \( \int_\infty^{-\infty} \frac{dz}{1+z^2} \) as \( R \) goes to \( \infty \). To see the second term goes to 0 as \( R \) goes to \( \infty \), we note:

\[ \left| \int_{c\infty} \frac{dz}{1+z^2} \right| \leq \int_0^{\pi} \left| \frac{iRe^{i\theta}}{1+R^2e^{2i\theta}} \right| d\theta \leq \int_0^{\pi} \frac{R}{R^2 - 1} d\theta \leq \frac{\pi R}{R^2 - 1} \to 0 \]
as \( R \) goes to \( \infty \).
1. (5 points) Remark: The solutions here are omitted since it’s sort of tedious computation. And the contours can always be chosen as \( \{ z : z \in \mathbb{R}, -R < z < R \} \cup \{ z : z = Re^{i\theta}, 0 \leq \theta \leq \pi \} \). The answers for the integrals are \( \pi/\sqrt{2}, 2\pi/3, \pi/3, \pi/2, \pi/2 \).

3. (5 points) Proof: As usual, we take the contour \( C \) as \( \{ z : z \in \mathbb{R}, -R < z < R \} \cup \{ z : z = Re^{i\theta}, 0 \leq \theta \leq \pi \} \), and the function \( f(z) = \frac{e^{iz}}{(z^2 + a^2)(z + bi)} \).

\( f(z) \) has singularities \( ai, -ai, bi, \) and \( bi \). Since the real parts of \( a \) and \( b \) are both positive, the singularities falling into \( C \) are \( ai \) and \( bi \). So,

\[
\int_C f(z)dz = 2\pi i \left[ \frac{e^{iz}}{(z^2 + a^2)(z + bi)} \bigg|_{z=bi} + \frac{e^{iz}}{(z^2 + b^2)(z + ai)} \bigg|_{z=ai} \right]
\]

We define \( \{ z : z \in \mathbb{R}, -R < z < R \} \) as II, and \( \{ z : z = Re^{i\theta}, 0 \leq \theta \leq \pi \} \) as I. And note

\[
\left| \int_I f(z)dz \right| = \left| \int_0^\pi \frac{e^{i(R \cos \theta + i \sin \theta)}}{(R^2 e^{2bi} + a^2)(R^2 e^{2bi} + b^2)} iRe^{i\theta} d\theta \right|
\]

\[
\leq \int_0^\pi \frac{Re^{-R \sin \theta}}{(R^2 - |a|^2)(R^2 - |b|^2)} d\theta \leq \frac{\pi}{(R^2 - |a|^2)(R^2 - |b|^2)} \frac{R}{R^2 - |b|^2}
\]

It’s clear that the last term goes to 0 as \( R \) goes to \( +\infty \). So, we finally get

\[
\frac{\pi}{a^2 - b^2} \left( \frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right) = \int_{-\infty}^{+\infty} \frac{e^{ix}}{(x^2 + a^2)(x^2 + b^2)} dx = \int_{-\infty}^{+\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx
\]

The last ”\( = \)” is because \( \frac{\sin x}{(x^2 + a^2)(x^2 + b^2)} \) is an odd function and it vanishes under the integration over the whole real line.

6. (9 points)

(a) If \( a \) and \( b \) are unequal complex numbers with positive real parts, prove

\[
\int_{-\infty}^{\infty} \frac{x \sin x dx}{(x^2 + a^2)(x^2 + b^2)} = \pi \left( \frac{e^{-a} - e^{-b}}{b^2 - a^2} \right)
\]

Proof: Let \( C_R \) be the semicircular contour \( z = Re^{i\theta}, 0 \leq \theta \leq \pi, \) and \( \pi = C_R \cup [-R, R] \). Then \((x^2 + a^2)(x^2 + b^2) = 0\) if and only \( x = ai, -ai, bi, \) or \(-bi\). Since \( Rea, Reb > 0 \), we have \( Im(ai), Im(bi) > 0 \). So only \( ai, bi \) fall inside \( C \) when \( R \) is large enough. Hence

\[
\int_{C} \frac{ze^{iz}dz}{(z^2 + a^2)(z^2 + b^2)} = 2\pi i[Res(f(ai) + Res(f(bi))] = \pi i \left( \frac{e^{-a} - e^{-b}}{b^2 - a^2} \right)
\]
Meanwhile
\[ \left| \int_{C_R} \frac{ze^{iz}dz}{(z^2 + a^2)(z^2 + b^2)} \right| \leq \int_0^\pi \frac{Rd\theta}{(R^2 - |a|^2)(R^2 - |b|^2)} \to 0 \]
as \( R \to \infty \). So let \( R \to \infty \), we get
\[ \int_{-\infty}^{\infty} \frac{xe^{ix}dx}{(x^2 + a^2)(x^2 + b^2)} = \pi i \frac{e^{-a} - e^{-b}}{b^2 - a^2} \]
Equating the imaginary part, we get
\[ \int_{-\infty}^{\infty} \frac{x \sin x dx}{(x^2 + a^2)(x^2 + b^2)} = \pi \frac{e^{-a} - e^{-b}}{b^2 - a^2} \]

(b) By l’Hospital’s rule, find the limit of the right-hand member as \( b \to a \) in part (a). Then determine whether this limit agrees with the value of the integral for \( b = a \).
Solution: By the same method as in part (a), we can find
\[ \int_{-\infty}^{\infty} \frac{x \sin x dx}{(x^2 + a^2)^2} = \pi e^{-a} \frac{1}{2a} \]
This is exactly the limit of \( \frac{\pi(e^{-a} - e^{-b})}{b^2 - a^2} \) as \( b \to a \), by l’Hospital rule.

(c) If \( f \) denotes the integrand in part (a), and \( I \) denotes the value of the integral, show that
\[ \left| \int_{-R}^{R} f(x)dx - I \right| \leq \frac{\pi R}{(R^2 - |a|^2)(R^2 - |b|^2)} \]
where \( R > \max(|a|, |b|) \).
Proof:
\[ \left| \int_{-R}^{R} f(x)dx - I \right| \leq \left| \int_C f(x)dx - I \right| + \left| \int_{C_R} \frac{ze^{iz}}{(z^2 + a^2)(z^2 + b^2)} |dz| \right| \leq \frac{R\pi}{(R^2 - |a|^2)(R^2 - |b|^2)} \]

Section 4.4

1. (3 points) Proof: Set \( R = e^t \). Then
\[ \frac{(\log R)^m}{R} = \frac{t^m}{e^t} < \frac{(m + 1)!}{t} \to 0 \]
as \( t \to \infty \). So \( \lim_{R \to \infty} \frac{(\log R)^m}{R} = 0 \) along \( R = e^t \forall m \in \{0, 1, 2, 3, \ldots\} \), and set 
\( \rho = R^{-1} \), then \( |\rho (\log \rho)^m| = \frac{(\log R)^m}{R} \). So \( \lim_{\rho \to 0^+} |\rho (\log \rho)^m| = \lim_{R \to \infty} \frac{(\log R)^m}{R} = 0 \). \( \square \)

### Additional Problems on Chapter 4

#### 4.1 (4 points) Proof: In \( \{ |z| \leq r \} \), \( f(z) \) can be written as \( \prod_{i=1}^{n} (z - a_i) h(z) \) where \( h \) is analytic in \( \{ |z| \leq r \} \). This shows \( g \) is analytic in \( \{ |z| \leq r \} \) except for removable singularities. Since \( r^2 / \bar{a}_i \) is outside \( \{ |z| = r \} \) \( i = 1, \ldots, n \) and \( h(z) \neq 0 \) in \( \{ |z| < r \} \), \( g(z) \) does not vanish in \( \{ |z| < r \} \). Finally \( |g(z)| = |f(z)| \) on \( \{ |z| = r \} \), since on \( |z| = r \), \( \left| \frac{r^2 - \bar{a}_i z}{r(t - a_i)} \right| = 1 \) by Chapter 1 problem 2.1, and \( |a_i / r| < 1, |z / r| = 1 \). \( \square \)

#### 4.2 (4 points) Proof: \( |g(0)| = |r^n f(0) / \prod_{i=1}^{n} a_i| \). Meanwhile by maximum principle, \( |g(0)| \leq \max_{|z| = r} |g(z)| = \max_{|z| = r} |f(z)| = M(r) \). So we get \( r^n / |a_1 \cdots a_n| \leq M(r) / |f(0)| \). \( \square \)

### Section 4.5

1. (6 points) Solution: Let \( D = \mathbb{C} \setminus [-\infty, \infty] \), then \( D \) is an analytic branch for \( z^{a-1} \). We use the same notation as in example 5.1. Then

\[
I = \int_C \frac{z^{a-1}}{1 - z} \, dz = -2\pi i
\]

Furthermore,

\[
I = \int_{R}^{\epsilon} \frac{r e^{i\pi} z^{a-1} e^{i\pi}}{1 + r} \, dr + \int_{R}^{\epsilon} \frac{(r e^{-i\pi}) z^{a-1}}{1 + r} e^{-i\pi} \, dr + J_1 + J_2
\]

In \( D \), \( z^{a-1} = e^{(a-1)(\log |z| + iarg z)} \) where \( -\pi < arg z < \pi \). So the first integral

\[
= e^{i\pi} e^{(a-1)(\log r + \pi)} / (1 + r) = e^{(p-1) \log r - \pi q} e^{(q \log r + \pi p) i} / (1 + r)
\]

We do similar thing to the second integral and get

\[
I = \int_{\epsilon}^{R} \frac{r^{p-1} e^{iq \log r} (-e^{-\pi q + \pi p i} + e^{\pi q - \pi p i})}{1 + r} \, dr + J_1 + J_2
\]

On the circle of radius \( R \) and \( \epsilon \), respectively,

\[
\left| \frac{z^{a-1}}{1 - z} \right| \leq \frac{R^{p-1} e^{\pi |q|}}{R - 1} \quad \left| \frac{z^{a-1}}{1 - z} \right| \leq \frac{\epsilon^{p-1} e^{\pi |q|}}{1 - \epsilon}
\]

Hence \( |J_1| \leq \frac{2\pi R e^{\pi |q|}}{R - 1} \), \( |J_2| \leq \frac{2\pi \epsilon e^{\pi |q|}}{1 - \epsilon} \). Since \( 0 < p < 1 \), by letting \( R \to \infty \) and \( \epsilon \to 0 \), we get

\[
\int_{0}^{\infty} \frac{r^{p-1} e^{iq \log r} (-e^{\pi q - \pi p i} + e^{-\pi q + \pi p i})}{1 + r} \, dr = -2\pi i
\]
By \( \sin \pi(p + iq) = \sin \pi p \cosh \pi q + i \cos \pi p \sinh \pi q \),

\[
\int_0^\infty \frac{t^{p-1}}{1 + r} e^{iq \log r} \, dr = \frac{2\pi i}{e^{-\pi q + \pi p} - e^{\pi q - \pi p}} = \frac{\pi}{\sin \pi(p + iq)} = \frac{\pi}{\sin \pi p \cosh \pi q + i \cos \pi p \sinh \pi q}
\]

Equating the real and imaginary parts, we get

\[
\int_0^\infty \frac{t^{p-1}}{t + 1} \cos(q \log t) \, dt = \frac{\pi \sin \pi p \cosh \pi q}{(\sin \pi p \cosh \pi q)^2 + (\cos \pi p \sinh \pi q)^2}
\]

\[
\int_0^\infty \frac{t^{p-1}}{t + 1} \sin(q \log t) \, dt = \frac{-\pi \cos \pi p \sinh \pi q}{(\sin \pi p \cosh \pi q)^2 + (\cos \pi p \sinh \pi q)^2}
\]

5. (6 points) Proof: Let \( f(z) = \sqrt{z} \log z/(1+z)^2 \). Let \( D = \mathbb{C}[0, \infty] \). Then \( D \) is an analytic branch of \( f(z) \). We let \( C \) be the contour enclosed by circles of radius \( R \) and radius \( \varepsilon \). Imitating example 5.1, we cut \( C \) into two parts, one part contains \(-1\), and the other one doesn’t. Then \( \int_C f(z) \, dz = 2\pi i \text{Res}(f, -1) \).

\( \text{Res}(f, -1) = (z^{1/2} \log z)|_{z=-1} = (z^{1/2}/z + z^{1/2} \log z/2z)|_{z=-1} = \pi/2 - i \). So

\[
\pi^2 i + 2\pi = \int_0^R \frac{\sqrt{x} \log x}{(1 + x)^2} \, dx - \int_0^R \frac{(xe^{2\pi i})^{1/2} \log(xe^{2\pi i})}{(1 + x)^2} \, dx + J_1 + J_2
\]

where \( J_1 \) and \( J_2 \) are the integration of \( f(z) \) along the circle of radius \( R \) and radius \( \varepsilon \), respectively. So

\[
\pi^2 i + 2\pi = \int_0^R \frac{\sqrt{x} \log x + \sqrt{x} (\log x + 2\pi i)}{(1 + x)^2} \, dx + J_1 + J_2
\]

On \( |z| = R, |f(z)| \leq \sqrt{R}(R + 2\pi)/(R - 1)^2 \) and on \( |z| = \varepsilon, |f(z)| \leq \sqrt{\varepsilon}(\log \varepsilon + 2\pi)/(1 - \varepsilon)^2 \). So let \( R \to \infty \) and \( \varepsilon \to 0 \), we get \( |J_1| \to 0 \) and \( |J_2| \to 0 \), respectively. Hence \( \pi^2 i + 2\pi = 2 \int_0^\infty \frac{\sqrt{x} \log x}{(1 + x)^2} \, dx + 2\pi i \int_0^\infty \frac{\sqrt{x}}{(1 + x)^2} \, dx \). Equating real parts and imaginary parts, we’re done.

\( \square \)