Section 3.7

2. (4 points) Proof: Since $f$ is analytic in $G$ and $f \neq 0$ in $G$, $1/f$ is analytic in $G$. By Theorem 7.5, $1/|f|$ cannot have a maximum value anywhere in $G$ unless $f$ is a constant. So $1/|f|$ assumes its maximum value on $\partial G$, i.e. $|f|$ assumes its minimum value on $\partial G$. \hfill $\square$

5. (6 points) Proof: Let $g = e^h$. Then $g$ is analytic in $D$ and is not constant. So $|g|$ doesn’t attain maximum in $D$. By Problem 2, $|g|$ doesn’t attain minimum in $D$ either. Since $|g| = e^{Reh}$, this means $Reh$ attains neither a maximum nor a minimum in $D$.

Let $f$ and $g$ be analytic in the bounded domain $D$. Set $h = f - g$. Then $Reh = 0$ on $\partial D$. By part one, this implies $h =$constant. So $f - g = ic$, for some constant $c$. \hfill $\square$

7. (6 points) Proof: Since $f$ is analytic for $|z| \leq R$, and $f(0) = 0$, we can write $f$ as $f(z) = zg(z)$, where $g(z)$ is analytic for $|z| \leq R$. On the boundary $\{|z| = R\}$, $|g(z)| = |f(z)/z| \leq M/R$. By maximum principle, $|g(z)| \leq M/R$ in $\{|z| < R\}$. So $|f(z)| \leq |g(z)||z| \leq |z|M/R < 1$ in $\{|z| < R\}$, unless $g =$ constant. \hfill $\square$

Section 3.8

1. (7 points) Solution: This kind of problem can be solved quite easily by looking at a function’s Laurent series. Unfortunately, the most useful theorem is not in section 8, but in section 9 (Theorem 9.4, page 167). Of course, straightforward observation is also beneficial in some cases.

(1) $e^z$: $e^z$ is analytic everywhere in $\mathbb{C}$. To judge the property of $\infty$, we consider all the three possibilities. First, $\infty$ cannot be removable by problem 3, since $e^z$ is not a constant function. Second, $\infty$ is not a pole, since $|e^{in}| \leq 1$, no matter how large $n \in \mathbb{N}$ is. So $\infty$ is an essential singularity.

(2) $\cos z/z$: Only 0 or $\infty$ could have problems since $\cos z/z$ is analytic elsewhere. Note the Laurent series of $\cos z/z = \sum_{n=2k}^{\infty} \frac{(iz)^n}{2^{2k}n!}$, where $k \in \mathbb{N} \cup \{0\}$, we conclude 0 is a pole by Theorem 9.4, since $1/z$ appears in the series. To see the property of $\infty$, replace $z$ with $1/\zeta$, it’s clear that $\zeta = 0$ is an essential singularity, by Theorem 9.4. Hence $\infty$ is an essential singularity of $\cos z/z$. 

1
(3) \( e^{z-1}/(z-1) \): 1 is a pole and 0 is removable. Replace \( z \) with 1/\( \zeta \), we get \( (e^{\zeta} - 1)/\zeta \). It's clear that this function is not differentiable at \( \zeta = 0 \) (argue by direct computation according to the definition of being analytic). So, \( \infty \) cannot be removable. Furthermore, if \( z \to \infty \) along the negative x-axis, then \( e^{z-1}/(z-1) \) goes to 0. So, \( \infty \) cannot be a pole. Hence, \( \infty \) has to be an essential singularity.

(4) \( z^2-1/z^2+1 \): This function is equal to \( 1 - \frac{2}{z^2+1} \). So, \( i \) and \( -i \) are two poles. Replace \( z \) with 1/\( \zeta \), we get \( 1 - \frac{2\zeta^2}{1+\zeta^2} \). This new function is differentiable at \( \zeta = 0 \). So, \( \infty \) is a removable singularity of the original function.

(5) \( z^3/z^4+2z^2 \): By similar argument, \( i \) and \( -i \) are two poles. 0 is a removable singularity. And \( \infty \) is also a pole, since after replacing \( z \) with 1/\( \zeta \), we get \( 1 - \frac{1}{\zeta+1} \).

(6) \( e^{\cosh z} \): \( \cosh z \) is an entire function, so is \( e^z \). Since the function under consideration is the composition of two entire functions, it’s entire. To judge \( \infty \), note first by problem 3, \( \infty \) cannot be removable. Let \( z = in \) where \( n \) is just a natural number. Then we can see, as \( n \to +\infty \), hence \( z \to \infty \), \( e^{\cosh z} \) is bounded. So, \( \infty \) cannot be a pole. So, \( \infty \) has to be an essential singularity.

(7) \( z(z-\pi)^2/\sin z \): We first solve the equation \( e^{iz} = e^{-iz} \) and get solutions \( z = k\pi \) where \( k \in \mathbb{Z} \). For \( k \neq 1, 0 \), \( k\pi \) becomes a pole since \( \sin z = 0 \) here. For 0, as \( z \to 0 \), \( \sin z \to 1 \), by the definition of the derivative of \( \sin z \) at 0. So, 0 is a pole. For \( \pi \), note \( \sin(z - \pi) = -\sin z \), we again return to the previous case. But this time the dominator \( \sin(z - \pi) \) and the nominator \( (z - \pi) \) have the same power. So, \( z(z-\pi)^2/\sin z \) \( \to \pi \) as \( z \to \pi \). Hence, \( \pi \) is a removable singularity. Let \( z \to \infty \) along the positive x-axis, the function has no limit. So, \( \infty \) cannot be a pole or removable. So, it’s an essential singularity.

3. (5 points) Proof: If a function is analytic in the extended plane, then in particular, it’s analytic at \( \infty \). So it must have a definite finite value at \( \infty \) and is continuous at \( \infty \). Hence, it is bounded in a neighbourhood of \( \infty \), say, \( \{ z : |z| > M \} \) for some positive number M. Meanwhile, this function is bounded in the closed disc \( \{ z : |z| \leq M \} \). So, this analytic function is bounded on the whole plane. By Liouville’s Theorem, it has to be a constant.

6. (6 points) Solution:

(i) \( e^z/z^5 = \sum_{n=-\infty}^{\infty} \frac{z^n}{(n+5)!} \). So \( a_n = \frac{1}{(n+5)!} \) and the principle part is \( \sum_{n=-5}^{-1} \frac{z^n}{(n+5)!} \).

(ii) \( \sin z/\sin^2 z = \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1} + 1}{2(n+2)!} \frac{z^{n+1}}{2(n+2)!} \). So \( b_n = \frac{(-1)^{n+1} + 1}{2(n+2)!} \frac{(-1)^{n+1}}{2} \) and the principle part is \( (z - 2\pi)^{-1} \).

(iii) \( 6/1-z^3 = -\sum_{n=-3}^{\infty} \frac{C_n^3(z-1)^n}{n+3} \). So \( c_n = -C_n^3 \) and principle part is \( -\sum_{n=-3}^{\infty} C_n^3(z-1)^n \).

Section 3.9
4. (4 points) Proof: Since \( f(z) \) is analytic for \( |z| \neq 0 \), we apply Theorem 9.2 to the case \( \alpha = 0 \) and get

\[
J_n(\omega) = \frac{1}{2\pi i} \int_{|z|=1} \frac{e^{\omega(z^{-1}/z)}}{z^{n+1}} dz
\]

\[
= \frac{1}{2\pi i} \int_0^{2\pi} e^{\omega(e^{i\theta}-e^{-i\theta})/2} e^{-(n+1)i\theta} i e^{i\theta} d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} e^{i\omega \sin \theta} e^{-n\theta} d\theta
\]

\[
= \frac{1}{2\pi} \left[ \int_0^{2\pi} \cos(\omega \sin \theta - n\theta) d\theta + \int_0^{2\pi} i \sin(\omega \sin \theta - n\theta) d\theta \right]
\]

\[
= \frac{1}{\pi} \int_0^{\pi} \cos(\omega \sin \theta - n\theta) d\theta
\]

5. (4 points) Proof:

\[
\frac{d^m}{d\omega^m} J_n(\omega) = \begin{cases} 
\frac{1}{\pi} \int_0^\pi \cos(\omega \sin \theta - n\theta) \sin^m \theta d\theta, & m = 0 \mod 4 \\
\frac{1}{\pi} \int_0^\pi -\sin(\omega \sin \theta - n\theta) \sin^m \theta d\theta, & m = 1 \mod 4 \\
\frac{1}{\pi} \int_0^\pi -\cos(\omega \sin \theta - n\theta) \sin^m \theta d\theta, & m = 2 \mod 4 \\
\frac{1}{\pi} \int_0^\pi \sin(\omega \sin \theta - n\theta) \sin^m \theta d\theta, & m = 3 \mod 4
\end{cases}
\]

So

\[
\left| \frac{d^m}{d\omega^m} J_n(\omega) \right|_{\omega=0} = \begin{cases} 
\frac{1}{\pi} \int_0^\pi \cos n\theta \sin^m \theta d\theta, & m = 0 \mod 2 \\
\frac{1}{\pi} \int_0^\pi -\sin n\theta \sin^m \theta d\theta, & m = 1 \mod 2
\end{cases}
\]

By Problem 2, we see this is 0 for 0 \( \leq m < n \). So \( J_n \) has a zero of order \( n \) at \( \omega = 0 \)

**Additional Problems on Chapter 3**

3.3 (4 points) Proof: Assume \( \max_{|z|=1} |1/z - f(z)| < 1 \). Then

\[
\left| \int_{|z|=1} [1/z - f(z)] dz \right| \leq \int_{|z|=1} |1/z - f(z)||dz| < 2\pi
\]

By Cauchy’s Theorem, \( \int_{|z|=1} [1/z - f(z)] dz = 2\pi i \). Contradiction.

4.2 (4 points) Proof:

\[
\frac{1}{2\pi i} \int_C \frac{P'(z)}{P(z)} dz = \frac{1}{2\pi i} \int_C \sum_{k=1}^{n} \frac{P(z)}{z - z_k} dz = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_C \frac{dz}{z - z_k} = \sum_{k=1}^{n} N(z_k)
\]

where \( N(z_k) \) is the winding number of \( z_k \) with respect to \( C \).