Section 2.1

2. (4 points) Solution: (a) Neither. (b) The second one. □

10. (4 points) Proof: Plug the expression of $\Delta \zeta$ into the first equation, we get

$$\Delta \omega = f'(\zeta)(g'(z)\Delta z + \varepsilon_2|\Delta z|) + \varepsilon_1|(g'(z)\Delta z + \varepsilon_2|\Delta z|)|$$

Since as $\Delta z \to 0$, $\Delta \zeta \to 0$, we conclude $\Delta z \to 0$ implies $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$. Hence

$$\Delta \omega = f'(g(z))g'(z)\Delta z + o(|\Delta z|)$$

Let $\Delta z \to 0$, we are done. □

Section 2.2

7. (6 points) Proof:

$$\cos(z_1 + z_2) + i \sin(z_1 + z_2)$$

$$= e^{z_1+z_2}$$

$$= e^{z_1}e^{z_2}$$

$$= (\cos z_1 + i \sin z_1)(\cos z_2 + i \sin z_2)$$

$$= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) + i(\cos z_1 \sin z_2 + \sin z_1 \cos z_2)$$

(1)

Replace $z_1$, $z_2$ with $-z_1$, $-z_2$ in the above equation, we get

$$\cos(z_1 + z_2) - i \sin(z_1 + z_2)$$

$$= (\cos z_1 \cos z_2 - \sin z_1 \sin z_2) - i(\cos z_1 \sin z_2 + \sin z_1 \cos z_2)$$

(2)

Divide (1)-(2) by $2i$, we get

$$\sin(z_1 + z_2) = \frac{2i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)}{2i} = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

Divide (1)+(2) by 2, we get

$$\cos(z_1 + z_2) = \frac{2(\cos z_1 \cos z_2 - \sin z_1 \sin z_2)}{2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$
13. (3 points) Proof:
\[
2\sqrt{2}e^{\pi i/12} \\
= 2\sqrt{2}e^{\pi i/3}e^{-\pi i/4} \\
= 2\sqrt{2}(1/2 + \sqrt{3}/2i)(\sqrt{2}/2 - \sqrt{2}/2i) \\
= (1 + \sqrt{3})(1 - i) \\
= (1 + \sqrt{3}) + (\sqrt{3} - 1)i
\]

15. (4 points) Proof: \( \omega = e^{(1+i)t} = e^t e^{it} \). So, when \( t \) goes from \(-\infty \) to \( \infty \), \( \text{Im}\omega = e^{it} \) repeats the value on the unit circle centered at origin, and \( \text{Re}\omega = e^t \) gets larger and larger. So the graph of \( \omega \) is a logarithmic spiral.

Section 2.3

2. (3 points) Proof: Since \( e^{\log 1 + i\pi/2} = e^{\pi i/2} = i \) and \( 0 \leq \pi/2 < 2\pi \), we get by definition \( \log i = \pi i/2 \). Hence
\[
i^i = e^{i\log i} = e^{i(\pi i/2 + i2\pi k)} = e^{-(4k+1)\pi/2}
\]
where \( k \in \mathbb{Z} \).

7. (12 points) Solution:
(i) \( \omega = \cos^{-1} z \) implies \( \cos\omega = z \), i.e. \( (e^{i\omega} + e^{-i\omega})/2 = z \). Hence \( e^{2i\omega} - 2ze^{i\omega} + 1 = 0 \). Solve this equation, we get
\[
e^{i\omega} = \frac{2z + \sqrt{4z^2 - 4}}{2} = z + \sqrt{z^2 - 1}
\]
Note here square roots are allowed to take two values, so it suffices to take only one of the roots of the quadratic equation. Since \( [z - (z^2 - 1)^{1/2}][z + (z^2 - 1)^{1/2}] = 1 \), we conclude \( z + (z^2 - 1)^{1/2} \neq 0 \). So \( \log[z + (z^2 - 1)^{1/2}] \) is well defined and \( w = -i \log[z + (z^2 - 1)^{1/2}] \).
(ii) \( \omega = \tan^{-1} z \) implies
\[
z = \tan\omega = -i\frac{e^{i\omega} - e^{-i\omega}}{e^{i\omega} + e^{-i\omega}}
\]
So we get
\[
e^{2i\omega} = \frac{i - z}{i + z}
\]
Take logarithm, we get
\[ \omega = \frac{i}{2} \log \frac{i + z}{i - z} \]

(iii) \( \omega = \cosh^{-1} z \) implies
\[ z = \frac{e^{i\omega} + e^{-i\omega}}{2} = \frac{e^\omega + e^{-\omega}}{2} \]

So \( e^{2\omega} - 2ze^\omega + 1 = 0 \). Solve this equation and take only one of the roots since square roots are allowed to take two values

\[ e^\omega = z + \sqrt{z^2 - 1} \]

So we get \( e^{2\omega} = z + (z^2 - 1)^{1/2} \).

(iv) \( \omega = \tanh^{-1} z \) implies \( z = \frac{e^\omega - e^{-\omega}}{e^\omega + e^{-\omega}} \). So \( z(e^{2\omega} + 1) = e^{2\omega} - 1 \). Simplify this equation, we get \( e^{2\omega} = \frac{1 + z}{1 - z} \). Hence \( \omega = \frac{1}{2} \log \frac{1 + z}{1 - z} \).

8. (4 points) Proof: Since \( \tan^{-1} z = \frac{i}{2} \log \frac{1 + z}{1 - z} \),

\[ \frac{d}{dz} \tan^{-1} z = \frac{i - z}{2i + z} \left( \frac{i - z}{2} \right) = \frac{1}{z^2 + 1} \]

Since \( \sinh^{-1} z = \log[z + (z^2 + 1)^{1/2}] \), we get

\[ \frac{d}{dz} \sinh^{-1} z = \frac{1 + \frac{1}{2}(z^2 + 1)^{-1/2}2z}{z + (z^2 + 1)^{1/2}} = \frac{1 + \frac{z}{\sqrt{z^2 + 1}}}{z + \sqrt{z^2 + 1}} = \frac{1}{\sqrt{z^2 + 1}} \]

Chapter 2 Additional Problems

1.3 (4 points) Proof: We note the following relations

\[ \sum_{i=1}^{n} \alpha_i = \frac{\text{the coefficient of } z^{n-1} \text{ in } P(z)}{-1 \times \text{the coefficient of } z^n \text{ in } P(z)} \]

and

\[ \sum_{j=1}^{n-1} \beta_i = \frac{\text{the coefficient of } z^{n-2} \text{ in } P'(z)}{-1 \times \text{the coefficient of } z^{n-1} \text{ in } P'(z)} \]

Since the coefficient of \( z^{n-2} \) in \( P'(z) = (n - 1) \times \text{the coefficient of } z^{n-1} \text{ in } P(z) \), and the coefficient of \( z^{n-1} \) in \( P'(z) = n \times \text{the coefficient of } z^n \text{ in } P(z) \), we can conclude the desired equation.

1.4 (6 points)
(a) Solution: \( \text{Re}(\alpha z + \beta) \) has the form of \( ax + by + c \) where \( x \) and \( y \) are the real and imaginary parts of \( z \), respectively. So \( \text{Re}(\alpha z + \beta) < 0 \) if and only if \( ax + by + c < 0 \) and therefore stands for a half plane.

(b) Proof: \( z \) is a root of \( Q(z) \) if and only if \( \bar{z} \) is a root of \( P(\bar{z}) \). So the zeros of \( Q \) lie in the half plane \( \text{Re}(\alpha z + \beta) < 0 \) if and only if the zeros of \( P \) have negative real parts. By Example 1.3, all the zeros of \( P'(\bar{z}) \) also have negative real parts. But \( P'(\bar{z}) = Q'(z) \frac{dz}{d\bar{z}} \). So \( \bar{z} \) is a root of \( P'(\bar{z}) \) if and only if \( z \) is a root of \( Q'(z) \). So by the correspondence between \( z \) and \( \bar{z} \), all the zeros of \( Q'(z) \) lie in the half place \( \text{Re}\bar{z} < 0 \).

(c) Proof: Clear from (b).