Section 1.4

2. (2 points) Solution: \( Rez > 1, 0 < |z| < 1, \text{Im}z < 2|z|, |z - 1| < |z + i| \) stand for domains. \(|z| \leq 1\) is not a domain, since it’s closed. And the last one, \(2|z^2 - 1| < 1\), is not, since it’s not connected (the existence of \(z^2\) in the formula makes the pre-image consist of two sheaves).

3. (3 points) Solution:
   (i) \(\omega = z^3\) maps \(\text{Arg} z\) to \(3\text{Arg}\, z\) mod \(2\pi\). In order that the \(\omega\) plane is just covered once, we should have \(0 \leq \theta < 2\pi/3\).
   (ii) \(\omega = z^4\) maps \(\text{Arg} z\) to \(4\text{Arg}\, z\) mod \(2\pi\). In order that the \(\omega\) plane is just covered once, we should have \(0 \leq \theta < \pi/2\).
   (iii) \(\omega = z^6\) maps \(\text{Arg} z\) to \(6\text{Arg}\, z\) mod \(2\pi\). In order that the \(\omega\) plane is just covered once, we should have \(0 \leq \theta < \pi/3\).

4. (6 points) Solution:
   (i) \(\omega = -z\): \(g(\omega) = -\omega, R = \mathbb{C}\).
   (ii) \(\omega = 1/z\): \(g(\omega) = 1/\omega, R = \mathbb{C} - \{0\}\).
   (iii) \(\omega = (1 - z)/(1 + z): g(\omega) = (1 - \omega)/(1 + \omega), R = \mathbb{C} - \{-1\}\).
   (iv) \(\omega = z^2: g(\omega) = \sqrt{\omega} e^{i\text{Arg} \omega/2}, R = \{z : 0 \leq \text{Arg} z < \pi\}\).
   (v) \(\omega = z^3: g(\omega) = |\omega|^{1/3} e^{i\text{Arg} \omega/3}, R = \{z : 0 \leq \text{Arg} z < 2\pi/3\}\).
   (vi) \(\omega = (z - 1)^3 + i: g(\omega) = |\omega - i|^{1/4} e^{i\text{Arg}(\omega - i)/4} + 1, R = \{z : 0 \leq \text{Arg}(z - 1) < \pi/2\}\).

Section 1.5

1. (5 points)
   (a) Solution: nowhere; 0; -1; roots of \(z^3 + 1 = 0\), i.e. \(-1, (1 - \sqrt{3}i)/2\) and \((1+\sqrt{3}i)/2\); roots of \(z^4 - 16 = 0\), i.e. \(-2, 2, 2i\) and \(-2i\); roots of \(z^8 + z^5 - z^4 - z = 0\), i.e. \(1, -1, i, -i, 0, (1 - \sqrt{3}i)/2\) and \((1 + \sqrt{3}i)/2\).
   (b) Solution: \((z + 1)/(z + 1)\) has a removable discontinuity at -1 and it should be defined 1 there to remove the discontinuity. \((z^4 - z^2)/(z^8 + z^5 - z^4 - z)\) has removable discontinuity at 0 and 1. To remove the discontinuity, it should be defined 0 and 1/4, respectively.
   (c) Solution: Replace \(z\) with \(1/\zeta\), we transform the above functions into the following ones: \(1/\zeta, \zeta, (1 + \zeta^2)/(\zeta + \zeta^2), 2\zeta^3/(1 + \zeta^3), (1 + \zeta)/(1 + \zeta),\)
(1 + 16\zeta^4)/(1 - 16\zeta^4), and \((\zeta^4 - \zeta^6)/(1 + \zeta^3 - \zeta^4 - \zeta^7)\). As \(\zeta \rightarrow 0\), their values, respectively, has no limit in \(C\); has limit 0; has no limit in \(C\); has limit 0; has limit 1; has limit 1; has limit 0.

5. (3 points) Solution: It seems reasonable to define \(a^{\sqrt{T}}\) as the product \(a \times a^{2/5} \times a^{1/100} \times a^{1/250} \times \ldots\). But difficulties arise. First, we need to know if the product is convergent. Second, equation \(z^n = z_0\) has \(n\) solutions and hence it’s not clear what value should be assigned to \(a^{1/5}\). Actually, as we shall see later, \(a^{\sqrt{T}}\) has infinitely many values.

6. (2 points) The proof is exactly what the hint suggests, and is very easy. So we skip it over.

Additional problems on chapter 1

1.2 (4 points) The proof is long, tedious, and easy. So I skip it over.

1.4 (5 points) Proof: We work by induction. For \(n=1\), we choose \(P_1(x) = x\). It’s clearly unique. Assume for \(n \leq m\), the claim is true. Then for \(n=m+1\), we have

\[
z^{m+1} + \frac{1}{z^{m+1}} - (z + \frac{1}{z})^{m+1} = - \sum_{k=1}^{m} C_{m+1}^k z^k \frac{1}{z^{m+1-k}}
\]

\[
= - \sum_{k=1}^{m} [C_{m+1}^k \frac{1}{z^{m+1-k}} + C_{m+1}^{m+1-k} \frac{1}{z^{m+1-k}}]
\]

\[
= \sum_{k=1}^{m} C_{m+1}^k [\frac{1}{z^{m+1-2k}} + z^{m+1-2k}]
\]

Since \(|m + 1 - 2k| \leq m\), by assumption, we have a unique polynomial \(Q_k\) such that \(1/z^{m+1-2k} + z^{m+1-2k} = Q_k(z + 1/z)\). Let \(P_{m+1}(x) = x^{m+1} - \sum C_{m+1}^k Q_k(x)\). Then this is the desired polynomial for \(n=m+1\). Uniqueness is shown in the proof as each \(Q_k\) is unique.

2.1 (3 points) Proof:

\[(1 - z)P(z) = (a_0 + a_1z + \cdots + a_nz^n) - (a_0z + a_1z^2 + \cdots + a_nz^{n+1})\]

\[= a_0 + (a_1 - a_0)z + \cdots + (a_n - a_{n-1})z^n - a_n z^{n+1}\]

So

\[|(1 - z)P(z)| > a_0 - [(a_0 - a_1)|z| + \cdots + (a_{n-1} - a_n)|z|^n - a_n z^{n+1}]\]

unless \(z = 0\). If \(|z| \leq 1\), then \(RHS \geq a_0 - [(a_0 - a_1) + \cdots + a_n] = 0\). So in this case, we have \(|P(z)| > 0\), which means \(P(z)\) can’t have roots in the closed unit disc centered at origin.
5.3 (7 points)
(a) Proof: Let $P(z) = \sum_{k=0}^{n} a_k z^k$ with $n \geq 1$, $a_n \neq 0$. Then $\lim_{|z| \to \infty} |P(z)| = \infty$. So outside some disc $\{z : |z| \leq R\}$, $|P(z)| > |a_0| = |P(0)|$. Therefore if the minimum of $|P(z)|$ for $|z| \leq R$ occurs at $z_0$, then $z = z_0$ gives the minimum of $|P(z)|$ with respect to the whole plane.

(b) Proof: It’s clear that $Q(z)$ is a polynomial. Since $Q(0) = 1$, we can see $Q(z)$ has the form of $1 + cz^m + \ldots$, with $m \geq 1$ and $\ldots$ are terms of higher degree.

(c) Proof: If $\alpha$ is a root of $ca^m = -1$, then $Q(\alpha z) = 1 + (\alpha z)^m + \ldots = 1 - z^m + \ldots$. By definition of $z_0$, $|Q(z)| \geq 1$. So, $|Q(\alpha z)|$ obtains its minimum modulus with respect to the plane at $z = 0$.

(d) Proof: let $z > 0$. Then as $z$ small enough, $|Q(\alpha z)|$ is close to $1 - z^m/2$, which is smaller than 1 for $z$ sufficiently small. Contradiction.