Section 5.1

1. (3 points) Proof: The images of the original straight lines under $\omega$ become, respectively, $t^2$ and $e^{2\phi}t^2$. Hence they are two straight lines through the origin meeting at an angle $2\phi$. So conformality needs not hold at points where $f'(z_0) = 0$. \hfill $\Box$

5. (10 points) Solution:
(a) $2z/(z + 1)$.
(b) $\frac{z-1}{z+1}i$.
(c) $z$.
(d) $iz$.
(e) $\frac{z+i}{z-1}$.

7. (3 points) Solution: $\omega$ maps $(-1, 0, 1)$ to three distinct points on the unit circle. So $\omega$ maps real axis to unit circle. Note $\omega(i) = 0$, so $\omega$ maps the upper half plane onto the unit circle. \hfill $\Box$

13. (3 points) Proof: Suppose $Tz = z$, then we get $cz^2 + dz = az + b$. This equation has two roots, if $c \neq 0$; only one root, if $c = 0$ and $d \neq a$; or infinitely many, if $c = 0$, $d = a$ and $b = 0$. The third case corresponds to identity mapping. So there are at most two fixed points unless $T$ reduces to the identity transformation. \hfill $\Box$

14 (5 points) Proof: Let $\omega = f(z)$ be the solution of $X(\omega, \omega_1, \omega_2, \omega_3) = X(z, z_1, z_2, z_3)$. Then clearly $f(z)$ is a bilinear transformation by explicit computation. And it satisfies $f(z_i) = \omega_i$, $i = 1, 2, 3$. If there’s another bilinear transformation $\omega = g(z)$ also satisfying $g(z_i) = \omega_i$, $i = 1, 2, 3$, then the bilinear transformation $f \circ g^{-1}(\omega)$ has three fix points. So $f \circ g^{-1} = id$. We conclude $f = g$. \hfill $\Box$

Section 5.2

10. (4 points) Proof: By solving equation $\omega = \frac{az+b}{cz+d}$, we find for any bilinear transformation $g$, it has an inverse and $g^{-1}$ is also a bilinear transformation. By explicit computation, we can also see the composition of two bilinear transformations is still a bilinear transformation. So the class of all
bilinear transformations form a group. Furthermore, if \( g \) maps \( \{|z| < 1\} \) onto itself, then \( g \) maps three distinct points on \(|z| = 1\) to three distinct points on \(|z| = 1\). If \( f \) is also such a mapping, \( f \circ g \) maps three distinct points on \(|z| = 1\) to three distinct points on \(|z| = 1\). So \( f \circ g \) is a bilinear transformation mapping \(|z| < 1\) onto itself. This shows the set of all bilinear mappings of \(|z| < 1\) onto itself forms a group.

11. (5 points) Proof: The first part of the problem has been proved in the proof of problem 10. For the second part, just follow the hint, which is detailed enough.

12. (5 points) Proof: For the first part, just imitate the proof for the second part of problem 11. For the second part, we have

\[
Im \frac{az + b}{cz + d} = Im \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz + d|^2} = Im \frac{adz + bc\bar{z}}{|cz + d|^2} = \frac{ad - bc}{|cz + d|^2}
\]

Hence \( Imz > 0 \) iff \( Im \frac{az+b}{cz+d} > 0 \). It shows this class of mappings maps the upper half-plane onto itself and also the lower half-plane onto itself.

Additional Problems on page 409

9.1 (6 points) Solution:
(i) \((z^{1/2})^2\) determines an entire analytic function since it equals to \(e^{\log z}\).
(ii) \((z^2)^{1/2}\) determines two entire functions, depending on the choice of square root.
(iii) \(\cos z^{1/2} = \frac{1}{2}(e^{i\sqrt{z}} + e^{-i\sqrt{z}})\) determines an entire function since \(\cos z = \cos(-z)\).
(iv) \((1 - z)^{1/2}\) is a three-valued analytic function.
(v) \((e^z)^{1/3}\) determines three entire functions, depending on the choice of cubic root.
(vi) \((\cos z)^{1/2}\) determines a two-valued analytic function.

9.3 (6 points) Proof: Since \( zg'(z) = f(z) \), it suffices to show that \(|z| = 1\) is a natural boundary for \( f \). For this, just follow the hint, which is detailed enough.