C1. Solution: Expand the left side of this equality into \[ \sum_{k=0}^{n} \binom{n}{k} (\cos \theta)^k (i \sin \theta)^{(n-k)}. \] Then compare the real parts and imaginary parts at both sides. □

C2. Proof: Let \( z = x + iy \) with \( x, y \) real numbers. Then, plug this representation into \( w = \frac{x^2 + y^2 - 1 + 2iy}{(x + 1)^2 + y^2} \), and rationalize the denominator, we get
\[
w = \frac{x^2 + y^2 - 1 + 2iy}{(x + 1)^2 + y^2}
\]
So, \( w \) is imaginary if and only if \( x^2 + y^2 = 1 \) holds. This is obviously equivalent to the condition that \( |z| = 1 \). Since \( w \) exactly stands for the angle determined by \( -1, z, 1 \), we are done. □

C3. Proof: The polar coordinate representation of \( 1 + i \) is \( \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}) \). So, by the formula in Problem 1 with \( \theta = \frac{\pi}{4} \), we get the desired equality.

C6. Solution:
\[
(1 + i)^2 = 2i, \quad \frac{3 + 4i}{1 - 2i} = -1 + 2i
\]
\[
z^3 = (x^3 - 3xy^2) + (3yx^2 - y^3)i, \quad \bar{z}z = x^2 + y^2
\]
\[ \frac{z}{\bar{z}} = \frac{x^2 - y^2 - 2xyi}{x^2 + y^2}, \quad \frac{z - i}{1 - i\bar{z}} = \frac{2x - 2xy + i[x^2 - (y - 1)^2]}{x^2 + (y - 1)^2}. \]

**C7.** Proof: Use the polar coordinate representation of $z$: $z = r \exp(i\alpha)$, we have $\Re(1/z) > 0$ if and only if $\Re\left(\frac{1}{r} \exp\{-i\alpha\}\right) > 0$, i.e. $\cos \alpha > 0$ and $\sin \alpha = 0$. So, $\alpha = 2k\pi$, for some integer $k$. By similar reasoning, we can find out $\Re z > 0$ if and only if $\alpha$ satisfies the above condition. So, $\Re \frac{1}{z} > 0$ if and only if $\Re z > 0$. \(\square\)

**C8.** Solution: To see what transformation is going on, we will regard $z$ as a point $(r, \alpha)$ in the plane, where $z = r \exp(i\alpha)$ is the polar coordinate representation of $z$.

(i) $z' = iz : (r, \alpha) \rightarrow (r, \alpha + \frac{\pi}{2})$. This is a counter clockwise rotation at the degree $\frac{\pi}{2}$.

(ii) $z' = 2z : (r, \alpha) \rightarrow (2r, \alpha)$. This is an expansion along the direction of $z$.

(iii) $z' = -z : (r, \alpha) \rightarrow (r, \alpha + \pi)$. This is a counter clockwise rotation at the degree $\pi$.

(iv) $z' = -2iz : (r, \alpha) \rightarrow (2r, \alpha - \frac{\pi}{2})$. So, this is a rotation combined with an expansion. \(\square\)

**C9.** Solution: Let $\alpha = 2 + 3i, \beta = 5 + 7i$. Then, by the fact $|\alpha\beta|^2 = |\alpha|^2|\beta|^2$, we have

\[ 962 = 13 \times 74 = |\alpha|^2 \times |\beta|^2 = |\alpha\beta|^2 = |11 + 29i|^2 = 11^2 + 29^2 \]

\(\square\)