MATH 321 Manifolds and Differential Forms (II)

Homework 9 Solution

Due November 15, 3:00 p.m.

6.4 (2 points) Solution: \( g = \log ||x||. \)

6.8 (6 points) Solution: A good way to do this problem is just observing the graphs of the parameterized curves, probably with the help of Mathematica. The winding numbers are 1, 0, 1 and 2, respectively. The following are the graphs of two of them.

Figure 1: Graph of the curve \( c(t) = (\cos^3 t, \sin^3 t) \), where \( t \in [0, 2\pi) \).

Figure 2: Graph of the curve \( c(t) = ((2 \cos t + 1) \cos t - 1/2, (2 \cos t + 1) \sin t) \), where \( t \in [0, 2\pi) \).

8.3 (4 points) Proof: Consider \( \mu(u + v, u + v) = 0 \). Expand the LHS by bilinearity, we get \( \text{LHS} = \mu(u, u + v) + \mu(v, u + v) = [\mu(u, u) + \mu(u, v)] + [\mu(v, u) + \mu(v, v)] = \mu(u, v) + \mu(v, u) \). For general case, the condition should be formulated in the following way: \( \mu(u_1, u_2, \ldots, u_n) = 0 \) if for some \( i \neq j \), \( u_i = u_j \). Then, by the multilinearity and same reasoning, we can show \( \mu(u_1, u_2, \ldots, u_n) = \text{sign}(\sigma) \mu(u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(n)}) \). For example, let us show \( \mu(u_1, u_2, \ldots, u_n) = -\mu(u_2, u_1, \ldots, u_n) \), we only need to expand the LHS of the equation \( \mu(u_1 + u_2, u_1 + u_2, u_3, \ldots, u_n) = 0 \).
8.5 (5 points) Proof: Let’s assume \( \mathbf{x} = (a_1, a_2, a_3)^T, \mathbf{y} = (b_1, b_2, b_3)^T \). Then

\[
\mathbf{x}^T \wedge \mathbf{y}^T = (a_1, a_2, a_3) \wedge (b_1, b_2, b_3) = (\sum_i a_i dx_i) \wedge (\sum_j b_j dx_j)
\]

\[
= (a_1 b_2 - a_2 b_1)dx_1 dx_2 + (a_2 b_3 - a_3 b_2)dx_2 dx_3 + (a_1 b_3 - a_3 b_1)dx_1 dx_3
\]

So,

\[
*(\mathbf{x}^T \wedge \mathbf{y}^T) = \begin{vmatrix} a_1 & b_1 & dx_1 dx_2 \\ a_2 & b_2 & dx_2 dx_3 \\ a_3 & b_3 & dx_1 dx_3 \end{vmatrix}
\]

This corresponds exactly to \( \mathbf{x} \times \mathbf{y} \).

8.9 (3 points) Proof:

\[
L^*(\lambda \mu)(v_1, v_2) = \lambda \mu(Lv_1, Lv_2)
\]

\[
= \det \begin{vmatrix} \lambda(Lv_1) & \lambda(Lv_2) \\ \mu(Lv_1) & \mu(Lv_2) \end{vmatrix}
\]

\[
= \det \begin{vmatrix} L^* \lambda(v_1) & L^* \lambda(v_2) \\ L^* \mu(Lv_1) & L^* \mu(Lv_2) \end{vmatrix}
\]

\[
= L^* \lambda L^* \mu(v_1, v_2)
\]