MATH 321 Manifolds and Differential Forms (II)

Homework 7 Solution

Due October 26, 3:00 p.m.

5.9 (4 points)
(i) Proof: By 5.8 (ii),

\[
\det A = \det A^T = \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(n),n} \\
= \sum_{\sigma \in S_n, \sigma(1)=1} \text{sign}(\sigma) a_{1,1} \cdots a_{\sigma(n),n} + \sum_{\sigma \in S_n, \sigma(1) \neq 1} \text{sign}(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(n),n}
\]

By condition, \( a_{\sigma(1),1} = 0 \) if \( \sigma(1) \neq 1 \). So

\[
\det A = a_{11} \sum_{\tau \in S_{n-1}} \text{sign}(\tau) a_{\tau(1),2} \cdots a_{\tau(n),n} = a_{11} \det A_{11}
\]

The last equality is by 5.8 (ii) again. \( \square \)

(ii) Proof: We let \( A = (a_{11}, a_{12}, \ldots, a_{n1}, a_{21}, a_{22}, \ldots, a_{n2}, \ldots, a_{1n}, a_{2n}, \ldots, a_{nn}) \) and \( e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T \) where \( 1 \) only appears in the \( j \)-th slot. Note \( a_j = (a_{1j}, \ldots, a_{nj})^T = \sum_i a_{ij} e_j \), we then by the multilinearity of the determinant function,

\[
\det A = \sum_j a_{ij} \det(a_{11}, \ldots, e_j, \ldots, a_{n1})
\]

\[
= \sum_j a_{ij} (-1)^{i+j} \begin{vmatrix} 1 & a_{i1} & \cdots & a_{i,j-1} & a_{i,j+1} & \cdots & a_{in} \\ 0 & A_{ij} & & & & & \end{vmatrix}
\]

\[
= \sum_j a_{ij} (-1)^{i+j} \det A_{ij}
\]

The last equality is by part (i). \( \square \)
5.10 (4 points) Proof: First observe that det $A$ is a polynomial of degree $n-1$ and if $x_i = x_j$ where $i \neq j$, then we have det $A = 0$. So det $A$ must have factors of the form $(x_i - x_j)$ where $i \neq j$. Hence det $A$ is the product of $\prod_{i \neq j}(x_i - x_j)$ and a polynomial $Q$. By counting the combinations, we see $Q$ has degree of 0. And finally it’s easy to see $Q$ is 1.

5.11 (4 points) Solution:

\[
P^* dx = \cos \phi \cos \theta r - r \sin \phi \cos \theta d\phi - r \cos \phi \sin \theta d\theta \]
\[
P^* dy = \cos \phi \sin \theta d\phi - r \sin \phi \sin \theta d\phi + r \cos \phi \cos \theta d\theta \]
\[
P^* dz = \sin \phi dr + r \cos \phi d\phi \]
\[
P^*(dx dy) = r \cos^2 \phi dr d\theta + r^2 \sin \phi \cos \phi d\theta d\phi \]
\[
P^*(dy dz) = r \sin \theta \phi dr + r \cos \phi \sin \theta d\phi \cos \theta d\theta d\phi + r^2 \cos^2 \phi \cos \theta d\theta d\phi \]
\[
P^*(dx dz) = r \cos \phi \cos^2 \phi + r \sin^2 \phi |dr d\phi| \cos \theta d\theta d\phi
\]

6.2 (4 points)

(i) Proof: $d \alpha = (d||x||^\alpha) \sum x_i dx_i = \alpha ||x||^{\alpha - 2} (\sum x_j dx_j) (\sum x_i dx_i) = \alpha ||x||^{\alpha - 2} \sum_{i \neq j} x_i x_j dx_i dx_j = \alpha ||x||^{\alpha - 2} \sum_{i < j} (x_i - x_j) x_i x_j dx_i dx_j = 0.$

(ii) Solution: Let $c(t) = (1 - t)x$ where $t \in [0, 1]$. Then $g(x) = \int_{c_x} \alpha = \int_0^1 ||(1 - t)x||^\alpha \sum_{i=1}^n (1 - t)x_i dx_i dt = -||x||^{2+\alpha} \int_0^1 (1 - t)^{\alpha + 1} dt$. So to let $g(x)$ be well-defined, $\alpha + 1 > -1$, i.e. $\alpha > -2$. And in this case, $g(x) = ||x||^{2+\alpha}/(\alpha + 2)$.

(iii) Proof: For $\alpha > -2$, $\int_0^1 (1 - t)^{\alpha + 1} dt = -1/(\alpha + 2)$, and $d||x||^{2+\alpha} = (\alpha + 2) ||x||^{\alpha} \sum x_i dx_i$. So $dg = \alpha$.

6.3 (4 points)

(i) Solution: $c(t) = tx$ where $t \in [1, \infty)$. Then $g(x) = \int_{c_x} ||x||^\alpha \sum_{i=1}^n x_i dx_i = \int_1^\infty t\alpha ||x||^\alpha \sum_{i=1}^n t x_i dx_i = ||x||^{\alpha + 2} \int_1^\infty t^{\alpha + 1} dt$. So let $g$ be well-defined, $\alpha + 1 < -1$, i.e. $\alpha < -2$, and hence $g(x) = -||x||^{\alpha + 2}/(\alpha + 2)$.

(ii) Solution: Tedious computation. (Omitted)

(iii) Solution: Let $\alpha = -3$, then $\alpha = ||x||^{-1} \sum x_i dx_i$ is Newton’s gravitational force, and $g(x) = 1/||x||$ is the corresponding potential energy.