MATH 321 Manifolds and Differential Forms
(II)

Homework 4 Solution

Due September 27, 3:00 p.m.

3.8. (4 points) Proof: By definition, \( x \in M \) is critical for \( g|_M \) if and only if \( dg(x) v = 0 \) for all \( v \in T_x M \), i.e. \( dg(x) v = 0 \) if \( df(x) v = 0 \) as \( T_x M = \ker df(x) \). Note \( df(x) = (\text{grad} f_1(x), \ldots, \text{grad} f_l(x)) \), so this is equivalent to \( dg(x) \perp v \) whenever \( v \perp \text{Span}\{\text{grad} f_1(x), \ldots, \text{grad} f_l(x)\} \). Hence \( \text{grad} g(x) = dg(x) \in \text{Span}\{\text{grad} f_1(x), \ldots, \text{grad} f_l(x)\} \). \( \Box \)

3.10. (6 points) (i) Solution:

\[
g(x) = x \cdot Ax = \sum_{i=1}^{n} x_i (Ax)_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} x_j
\]

where \( A = (a_{ij}) \). So

\[
\frac{\partial}{\partial x_k} g(x) = \sum_{j=1}^{n} a_{kj} x_j + \sum_{i=1}^{n} a_{ik} x_i = 2 \sum_{i=1}^{n} a_{ii} x_i
\]

as \( A = A^T \). Since \( \frac{\partial}{\partial x_k} g(x) = 2 (Ax)_k \), we get \( \text{grad} g(x) = 2Ax \). \( \Box \)

(ii) Proof: By definition, \( x \in M \) is a critical point of \( g|_M \) if and only if \( dg(x) v = 0 \) for all \( v \in T_x M \), i.e. \( dg(x) \perp T_x M \). Since \( M \) is a sphere, \( T_x M \perp = \text{Span}\{x\} \). Hence \( x \in M \) is a critical point of \( g|_M \) if and only if \( dg(x) = \lambda x \) for some \( \lambda \in \mathbb{R} \). By (i), \( dg(x) = 2Ax \). So \( Ax = \lambda x/2 \). This shows \( x \) is an eigenvector for \( A \). Obviously \( ||x|| = 1 \). \( \Box \)

(iii) Proof:

\[
g(x)x = (x \cdot Ax)x = x(Ax)^T x = xx^T A^T x = Ax
\]

by \( A = A^T \) and \( ||x|| = 1 \). So \( g(x) \) is the corresponding eigenvalue of \( x \). \( \Box \)

3.12. (5 points) Proof: \( (t - a \sin t)' = 1 - a \cos t \). Since \( a \in (0, 1) \), \( 1 - a \cos t > 0 \). So \( t - a \sin t \) is increasing. This implies \( f \) is i-1.

\( df(t) = (1 - a \cos t, a \sin t) \). So \( df(t) \) has rank 0 if and only if \( a \cos t = 1 \) and \( \sin t = 0 \). This is impossible since \( |a \cos t| \leq a < 1 \).

To see \( f^{-1} \) is continuous, it’s beneficial to look at the graph of \( f \). Then we can see from the graph that \( f^{-1} \) is continuous. (The graph is on the next page.) \( \Box \)
Figure 1: Graph of the curve \( f(t) = (t - a \sin t, 1 - a \cos t) \)

4.1. (5 points)

(i) Solution: \( d(e^{xy} \, dx) = -xye^{xy} \, dx \, dy - xy e^{xy} \, dx \, dz. \)

(ii) Solution:

\[
\begin{align*}
d(\sum_{i=1}^{n} x_i^2 \, dx_1 \ldots \hat{x_i} \ldots \, dx_n) &= \sum_{i=1}^{n} 2x_i \, dx_1 \, dx_1 \ldots \hat{x_i} \ldots dx_n \\
&= \sum_{i=1}^{n} (-1)^{i-1} 2x_i \, dx_1 \, dx_2 \ldots \, dx_n
\end{align*}
\]

(iii) Solution:

\[
\begin{align*}
d(||x||^p \sum_{i=1}^{n} (-1)^{i+1} x_i \, dx_1 \ldots \hat{x_i} \ldots \, dx_n) \\
&= \sum_{i=1}^{n} (-1)^{i+1} d(||x||^p) \, dx_1 \ldots \hat{x_i} \ldots dx_n \\
&= \sum_{i=1}^{n} (-1)^{i+1} (||x||^p + \frac{p}{2} x_i ||x||^{p-2} x_i) \, dx_1 \ldots \hat{x_i} \ldots dx_n \\
&= \sum_{i=1}^{n} (||x||^p + px_i^2 ||x||^{p-2}) \, dx_1 \ldots dx_i \ldots dx_n \\
&= (n + p) ||x||^p \, dx_1 \ldots dx_i \ldots dx_n
\end{align*}
\]

So this form is closed if and only if \( p = -n. \)