Math 7620, Spring 2009, Problem Set 1, Due February 2, 2009

For any set $X \subset \mathbb{E}^d$ and a function $f : X \to \mathbb{E}^d$ we say f is *congruence* if for all $\mathbf{p}, \mathbf{q} \in \mathbb{E}^d$, $|f(\mathbf{p}) - f(\mathbf{q})| = |\mathbf{p} - \mathbf{q}|$.

1. If $f : \mathbb{E}^d \to \mathbb{E}^d$ is a congruence and $f(\mathbf{0}) = \mathbf{0}$, show that for all $\mathbf{p}, \mathbf{q} \in \mathbb{E}^d$, $f(\mathbf{p}) \cdot f(\mathbf{q}) = \mathbf{p} \cdot \mathbf{q}$. (Hint: For all $\mathbf{p}, \mathbf{q} \in \mathbb{E}^d$, $\mathbf{p} \cdot \mathbf{q} = \frac{1}{2}(\mathbf{p}^2 + \mathbf{q}^2 - (\mathbf{p} - \mathbf{q})^2)$.)

2. If $f : \mathbb{E}^d \to \mathbb{E}^d$ is a congruence and $f(\mathbf{0}) = \mathbf{0}$, show that for all $\mathbf{p}, \mathbf{q} \in \mathbb{E}^d$ and scalars s and t, $(sf(\mathbf{p}) + tf(\mathbf{q}))^2 = (s\mathbf{p} + t\mathbf{q})^2$.

3. If $f : \mathbb{E}^d \to \mathbb{E}^d$ is a congruence, show that for all $\mathbf{p} \in \mathbb{E}^d$, $f(\mathbf{p}) = A\mathbf{p} + \mathbf{b}$, where $A = (A^{-1})^T$ is an orthogonal d-by-d matrix, and $\mathbf{b} \in \mathbb{E}^d$ is constant. $(\cdot)^T$ is the transpose.

4. For any set $X \subset \mathbb{E}^d$ and a congruence $f : X \to \mathbb{E}^d$, show that f extends to a congruence of all $\mathbb{E}^d$. Furthermore, the extension is unique when restricted to the affine span of X. (The *affine span of a set* is the intersection of all the affine linear subspaces that contain X. An affine linear subspace of $\mathbb{E}^d$ is the translate of a linear subspace.) When the affine span of $X \subset \mathbb{E}^d$ is $(d-1)$-dimensional there are precisely two extensions of f to $\mathbb{E}^d$.

5. Let X be any compact convex subset of $\mathbb{E}^d$, and let $f : \mathbb{E}^d \to X$ denote projection onto the nearest point on X. Show that f is uniquely defined and a *contraction*. In other words for all $\mathbf{p}, \mathbf{q} \in \mathbb{E}^d$, $|f(\mathbf{p}) - f(\mathbf{q})| \leq |\mathbf{p} - \mathbf{q}|$.