Homework 6

(1) Let \( \{B_t\}_{t \geq 0} \) be a standard Brownian motion and let \( a < 0 < b \). Define \( T(x, y) = \inf \{ t \geq 0 : B_t \in \{x, y\} \} \). Show that \( E[T(a, b)] = a^2 E[T(1, \frac{b}{a})] \), hence the expected exit time from a symmetric interval \([-b, b]\) is a constant multiple of \( b^2 \).

(2) Suppose that \( \{B_t\}_{t \in [0,T]} \) is a Brownian motion on \([0, T] \). Show that the time reversed process \( \{B_{T-t} - B_T\}_{t \in [0,T]} \) is a standard Brownian motion on \([0, T] \).

(3) Using the Lévy construction of Brownian motion given in class, show that if \( f : [0, 1] \to \mathbb{R} \) is continuous with \( f(0) = 0 \) and \( \{B(t) : t \geq 0\} \) is a standard Brownian motion, then for any \( \varepsilon > 0 \),
\[
P \left( \sup_{0 \leq t \leq 1} |B(t) - f(t)| < \varepsilon \right) > 0.
\]

(4) Let \( \{B_t\}_{t \geq 0} \) be a standard Brownian motion. Prove that
\[
\sup_{0 \leq s < t \leq 1} \frac{|B_t - B_s|}{|t - s|} = \infty \text{ a.s.}
\]
whenever \( \gamma \geq \frac{1}{2} \).

(5) Consider a (not necessarily nested) sequence of partitions \( 0 = t_0^{(n)} \leq t_1^{(n)} \ldots \leq t_{k(n)}^{(n)} = t \) with mesh converging to 0.

(a) Show that
\[
\lim_{n \to \infty} \sum_{j=1}^{k(n)} \left( B\left( t_j^{(n)} \right) - B\left( t_{j-1}^{(n)} \right) \right)^2 \to t \text{ in } L^2.
\]
(We say that Brownian motion has quadratic variation \( V_B^{(2)}(t) = t \).)

(b) Show that if the sequence of partitions satisfies \( \sum_{n=1}^{\infty} \sum_{j=0}^{k(n)} \left( t_j^{(n)} - t_{j-1}^{(n)} \right)^2 < \infty \), then the convergence in part (a) is almost sure.
(An example is partitioning \([0,1]\) with \( t_j^{(n)} = \frac{j}{2^n}, \text{ } j = 0, 1, ..., k(n) = 2^n \).)

(c) Argue that \( \sum_{j=1}^{k(n)} \left| B\left( t_j^{(n)} \right) - B\left( t_{j-1}^{(n)} \right) \right| \to \infty \text{ a.s.} \)
(6) A standard Brownian bridge is a Gaussian process \( \{ X(t) : 0 \leq t \leq 1 \} \) with continuous paths, mean 0 and covariance \( \text{Cov}(X(s), X(t)) = s(1-t) \) for \( 0 \leq s \leq t \leq 1 \). If \( \{ B(t) : t \geq 0 \} \) is a standard Brownian motion, verify that the following processes are Brownian bridges.

(a) \( X_1(t) = B(t) - tB(1) \)

(b) \( X_2(t) = (1-t)B \left( \frac{t}{1-t} \right) 1_{[0,1)}(t) \)

(7) Let \( \{ B(t) : t \geq 0 \} \) be a standard Brownian motion and let \( T \) be a stopping time with \( E[T] < \infty \). Define a sequence of stopping times by \( T_1 = T \), \( T_n = T(B_n) + T_{n-1} \) where \( T(B_n) \) is the same function as \( T \) but associated with the Brownian motion \( B_n(t) = B(t + T_{n-1}) - B(T_{n-1}) \).

(a) Show that \( \lim_{n \to \infty} \frac{B(T_n)}{n} = 0 \) a.s.

(b) Show that \( B(T) \) is integrable.

(c) Show that \( \lim_{n \to \infty} \frac{B(T_n)}{n} = E[B(T)] \) (which, combined with (a), gives Wald’s lemma)