(1) Let $X_n$ be the position of simple random walk on $\mathbb{Z}$ at time $n$. That is, $X_n = \sum_{k=1}^n \xi_k$ where $\xi_1, \xi_2, \ldots$ are i.i.d. with $P(\xi_1 = 1) = P(\xi_1 = -1) = \frac{1}{2}$. Show that $M_n = X_n^3 - 3nX_n$ is a martingale w.r.t. $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$.

(2) Suppose that $\xi_1, \xi_2, \ldots$ are independent with $E[\xi_i] = 0$ and $E[\xi_i^2] = \sigma_i^2 < \infty$, and set $S_n = \sum_{i=1}^n \xi_i$. Let $s_n^2 = \sum_{i=1}^n \sigma_i^2$. Show that $S_n^2 - s_n^2$ is a martingale w.r.t. $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$.

(3) Give an example of a (nonconstant) submartingale $X_n$. Suppose that $\xi, \sigma_i > 0$.

(4) Let $(\Omega, \mathcal{F}, P)$ be $[0, 1]$ with the Borel sets and Lebesgue measure. Let $\mathcal{F}_n = \sigma\left(\{[i-1, i) : j = 1, \ldots, 2^n]\right)$ and define $X_n = 2^n 1_{[0,2^{-n})}$. Show that $\{X_n\}$ is a (nonnegative) martingale w.r.t. $\{\mathcal{F}_n\}$. Does $X_n$ converge in $L^1$?

(5) Suppose $X_n^1$ and $X_n^2$ are supermartingales with respect to $\mathcal{F}_n$, and $N$ is a stopping time with $X_N^1 \geq X_N^2$. Show that $Y_n = X_n^1 1_{\{N > n\}} + X_n^2 1_{\{N \leq n\}}$ is a supermartingale.

(6) Let $X_n$ be a martingale with $X_0 = 0$ and $E[X_n^2] < \infty$. Show that for all $\lambda \geq 0$

$$P\left(\max_{1 \leq m \leq n} X_m \geq \lambda\right) \leq \frac{E[X_n^2]}{E[X_n^2] + \lambda^2}.$$  

(Hint: For any $c \in \mathbb{R}$, $(X_n + c)^2$ is a submartingale.)

(7) Let $\varphi \geq 0$ be any function with $\frac{\varphi(x)}{x} \to \infty$ as $x \to \infty$. Show that $E[\varphi(|X_i|)] \leq C$ for all $i \in I$ implies $\{X_i\}_{i \in I}$ is uniformly integrable.

(8) Let $\xi_1, \xi_2, \ldots$ be i.i.d. with $P(\xi_1 = 1) = P(\xi_1 = -1) = \frac{1}{2}$, and set $X_n = \sum_{k=1}^n \frac{\xi_k}{k}$. Show that $X_n$ converges to an integrable random variable $X$ with probability one. In other words, the random harmonic series is a.s. convergent.

(9) Suppose that $X_1, X_2, \ldots$ are i.i.d. picks from a density $f$ which is either equal to $f_0$ or $f_1$, both of which are strictly positive on $\mathbb{R}$. Show that under the null hypothesis $f = f_0$, the test statistic $A_n = \prod_{i=1}^n \frac{f_1(X_n)}{f_0(X_n)}$ converges a.s. as $n \to \infty$.

(10) Let $Z_1, Z_2, \ldots$ be i.i.d. standard normals, and let $\theta$ be an independent random variable with finite mean. Let $Y_n = Z_n + \theta$. In statistical terms, we have a sample from a normal population with unknown mean. The distribution of $\theta$ is called the prior distribution and $P(\theta \in \cdot | Y_1, \ldots, Y_n)$ is called the posterior distribution after $n$ observations. Show that $E[\theta | Y_1, \ldots, Y_n] \to \theta$ a.s. (The Bayes estimate is consistent.)