21. Prove that the Lie algebra $TR(n)$ of all $n \times n$ upper triangular matrices is solvable by establishing that $L_k = \{ A \in TR(n) : a_{ij} = 0 \text{ for } i + k > j \}$, $k = 0, 1, 2, \ldots$ are ideals and the quotient algebras $L_k/L_{k-1}$ are commutative.

22. Prove that, if $L$ is a simple Euclidean Lie algebra, then $[L] = L + iL$ is also simple. Deduce from here that $SL(n, \mathbb{C})$ and $SO(n, \mathbb{C})$ are simple Lie groups.

23. Let $[L]$ be the complexification of a Euclidean Lie algebra $L$. The Cartan involution $\theta$ of $[L]$ is defined by the formula $\theta(x + iy) = x - iy$ for $x, y \in L$. Prove that:

A. $\theta[z_1, z_2] = [\theta(z_1), \theta(z_2)]$ and $\theta(c_1z_1 + c_2z_2) = c_1\theta(z_1) + c_2\theta(z_2)$ for all $z_1, z_2 \in [L]$ and all real $c_1, c_2$.

B. Put $\theta(z, z) = (z, \theta(z)) = -k(z, \theta(z))$ where $k$ is the Killing form. Prove that $\theta(z, z) > 0$ if $z \neq 0$.

24. Prove the following properties of root system $\Sigma$.

A. $\theta(E_\alpha) \subset E_{-\alpha}$ and $-\alpha \in \Sigma$ if $\alpha \in \Sigma$.

B. $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})\alpha$.

Hint. Prove that $\theta(h, h) = 0$ for $h = [e_\alpha, e_{-\alpha}] - (e_\alpha, e_{-\alpha})\alpha$.

C. If $L$ is semisimple, then span($\Sigma$) coincides with the Cartan subalgebra $H$.

Hint. Prove that $h = 0$ if $h \in H$ and $(h, \alpha) = 0$ for all $\alpha \in \Sigma$.

D. If $\alpha \in \Sigma$, then $k\alpha \notin \Sigma$ for $k \neq \pm 1$ and $\dim(E_\alpha) = 1$.

Hint. Take $e_\alpha \in E_\alpha, e_{-\alpha} \in E_{-\alpha}$ such that $[e_\alpha, e_{-\alpha}] = 1$ and put $A_\alpha = ad(e_\alpha), A_{-\alpha} = ad(e_{-\alpha}), H_\alpha = ad(\alpha)$.

Note that $tr_M(H_\alpha) = 0$ in any subspace $M$ of $[L]$ invariant relative to $A_\alpha$ and $A_{-\alpha}$ and apply this to $M =$span($e_{-\alpha}, H, E_\alpha, E_{2\alpha}, \ldots, E_{k\alpha}, \ldots$).

25. Describe Dynkin diagrams of all sets $\Gamma$ of vectors in $\mathbb{R}^n$ with the properties:

1. $\alpha, \beta \in \Gamma$ and $\alpha \neq \beta$, then $\frac{2(\alpha, \beta)}{(\alpha, \alpha)}$ is a non-positive integer;

2. $\Gamma$ cannot be split into two orthogonal subsets.  \footnote{Linear independence is not required.}