L.S. Pontryagin, Topological Groups


The basic method of the present paragraph will be the expansion of various functions in Taylor series up to terms of degree two or, sometimes, three. The study of the Taylor coefficients that thus appear will lead us to the structure constants as well as to the relations they satisfy. It would have been possible, to be sure, to conduct the investigation in terms of derivatives, as is customary, but the use of Taylor series seems to me to be more fitting.

A) The remainders of series will not be written out in detail but will rather be indicated by ε's with various indices. In each case the order of vanishing of the remainder ε will be specifically stated. If ϵ is a function of the arguments x₁, x₂, x₃ then we shall say that ϵ is small of order q + 1 with respect to these arguments if ϵ/ρ^q goes to zero with ρ where ρ = √(1 + x₁^2 + x₂^2). ¹/₂

B) Just as in Chapter 3, whenever some letter is used to denote a point or a vector the same letter equipped with appropriate superscripts will denote the coordinates of that point or vector. This convention will be adhered to throughout the chapter and consequently need not be restated each time it is used.

Definition 47: Let G be an r-dimensional local Lie group and let D be a differentiable coordinate system in G (see Definition 39). If x and y are elements of G sufficiently close to the identity e then the product f = xy = f(x, y) is also close to e and the rule of multiplication may be written in coordinate form in terms of the coordinate system D:

\[ f = f(x, y) = f(x^1, x^2, x^3, y^1, y^2, y^3). \]

Since the coordinates of e are all 0's we have the special relations

\[ f(e, x) = f(x^1, x^2, x^3; 0, 0, 0) = x^1, \]

\[ f(e, y) = f(0, 0, 0; y^1, y^2, y^3) = y^1. \]

Since the function f is three times continuously differentiable they can be expanded in Taylor series up to terms of degree three. Moreover, because of (2) and (3), these expansions assume a somewhat special form; indeed it may readily be verified that

\[ f^i = x^i + y^i + a_{jk}^i x^j y^k + b_{jk}^i x^j x^k y^l + c_{jk}^i x^j y^k + \epsilon_s^i, \]  

(4)

where ε_s^i is small of degree four with respect to the coordinates of x and y. The numbers

\[ c_{jk}^i = a_{jk}^i - a_{kj}^i \]

are called the structure constants of G in the coordinate system D. The structure constants clearly satisfy

\[ c_{jk}^i = -c_{kj}^i \]

(6)

Relation (4) shows that any Lie group is, in the first approximation, commutative and isomorphic with a vector group, and that the departure from commutativity first appears in the second degree approximation. It is not difficult to show that even if G is commutative there may exist coordinate systems in which the second degree terms in (4) do not vanish. However, in the case of a commutative group we clearly have a_{jk}^i = a_{kj}^i, so that the structure constants all vanish. This fact constitutes the first hint of the great importance of the structure constants. Later it will be seen that, in fact, they determine the local structure of the group completely; it is this that justifies their name.

We now give an alternative definition of the structure constants which sheds further light on their role.

C) Let x and y be elements of G and consider the commutator q(x, y) (see Section 4, C))

\[ q = xyx^{-1}y^{-1} = q(x, y). \]

(7)

It turns out that in coordinate form (7) assumes the form

\[ q^i = c_{jk}^i x^j y^k + \epsilon_s^i \]

(8)

where ε_s^i is small of degree three with respect to the coordinates of x and y. Thus (8) may also be used to define the structure constants. Note that according to (4), (5) and (8) we have

\[ q^i(x, y) = f^i(x, y) - f^i(y, x) + \epsilon_s^i \]

(9)

where ε_s^i of small of degree three.

In order to verify (8) we first use (4) to obtain a second degree approximation to the coordinates of the element x⁻¹
to \( z \). A straightforward computation shows that if \( zz' = e \) then
\[
z'' = z^i + a_{jk}^i z^j z^k + \varepsilon_4^i \tag{10}
\]
Now letting \( z^* = xy \) and \( z = yx \), we have \( q = z^* z' \), and consequently, employing (4) once again along with (5) and (10), we obtain
\[
q^i = (x^i + y^i + a_{jk}^i x^j y^k) + (-x^i - y^i - a_{ijk} x^j y^k x^k)
+ a_{jk}^i(x^i + y^i)(x^j + y^j x^k) - a_{jk}^i(x^i + y^i)(x^j + y^j x^k) + \varepsilon_4^i
= c_{jk}^i x^j y^k + \varepsilon_4^i.
\]
Thus (8) is verified.

Theorem 81: The structure constants of any Lie group \( G \) satisfy the following relations:
\[
c_{ij}^p = -c_{ji}^p, \tag{11}
\]
\[
c_{ij}^p c_{kp}^q + c_{jp}^p c_{ki}^q + c_{kp}^q c_{ij}^p = 0. \tag{12}
\]
(Identity (12), sometimes known as the Jacobi identity, is intimately connected with the associativity of \( G \)).
Lie groups in physics

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Lie groups made occasional appearances in physics before the invention of quantum mechanics. However, Lie groups became a standard component of the theoretical physicist's tool kit only after the very prompt recognition by Hermann Weyl and Eugene Wigner in the 1920s that a knowledge of the rotation group (and after the invention of the Dirac equation, of the Lorentz group), would be useful in quantum mechanics. Widespread application of larger compact Lie groups began in the late 1950s when it was realized, first by Gell-Mann and Schwinger, that the ever growing families of "elementary" particles might be classified into representations of compact Lie groups. The correct model was invented by Gell-Mann and Ne'eman independently in 1961, and is now called flavor SU(3). A related, though somewhat distinct, and as it turned out far more important application of Lie groups stemmed from the invention by Yang and Mills of non-Abelian gauge theories, which generalize the U(1) invariance of quantum electrodynamics.

It is now understood that the strong, the electromagnetic and the weak interactions are dictated by a SU(3)xSU(2)xU(1) gauge symmetry.

Physicists, facing the need to explore ever more complicated Lie groups, during this period learned that Eugene Dynkin's ingenious techniques for analyzing Lie groups and algebras yielded great technical and conceptual simplification, and as a consequence, these tools took on a role somewhat similar to that of Feynman diagrams in quantum field theory. The success of the electroweak gauge theory in the 1970s led to attempts to unify the electroweak interaction with the strong interaction, and this required a systematic exploration of all gauge groups built on compact Lie groups, which saw widespread use of Dynkin diagrams. The recognition that string theory offered the first, and thus far only, possibility of uniting gravitation with the other three fundamental interactions into a consistent quantum theory has led to a much broader exploration of group-theoretic concepts. In particular, the gauge group symmetries in string theory are realized with underlying Kac-Moody, or infinite dimensional Lie, algebras, in which there is a pervasive use of Dynkin's techniques and concepts. The following references are only the tip of a continually growing iceberg.
Lie groups in physics

References

Dynkin Diagrams in the Physics of Particles, Fields and Strings

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1. Background: "The Group Pest" in the First Half of the Twentieth Century

In the introduction\(^1\) to the Second Edition of his classic "Group Theory and Quantum Mechanics", Herman Weyl feels sorry for the disappointment of the Physics community, whose members had surmised (and hoped) that "the group pest" would soon go away: "It has been rumored that the "group pest" is gradually cut out of physics. This is not true, in so far as the rotation and Lorentz groups are concerned." And yet, the incursion of Lie algebras in Physics before 1950 had been rather limited, mostly restricted indeed - as noted by Weyl - to the Lorentz group. There was no going back from that, after the 1905 revolutionary discovery of Special Relativity by Einstein, building on the cumulative results of Lorentz, Fitzgerald and Poincaré, but daring to cross into the realm of a less intuitive reality. In atoms - and then in nuclei - the angular momentum SO(3) subgroup of the Lorentz \( SO(1,3) \) - and soon its double-covering, \( Spin(3) = SO(3) = SU(2) \subset SO(1,3) = SL(2,C) \) - as noted by Weyl, these were the main algebraic tools exploited, with the physically very interesting (but algebraically modest) exception of Pauli's use\(^2\) of the dynamical SO(4) symmetry (i.e. energy degeneracy) of the Kepler problem to solve the quantum mechanical hydrogen atom problem, applying the methods of matrix mechanics. For the Lorentz and Poincaré groups (the latter group taking up the key role in the Hilbert space of Quantum Mechanics) the representations were indeed soon constructed and classified by Gelfand /Neumark and by Wigner, respectively, with Dirac's work - further extended by Bargmann and Wigner - providing the relativistic equations. The advent of General Relativity brought in the first case of a gauge symmetry, using modern particle physics terminology, i.e. the Principle of Covariance, or invariance under local diffeomorphisms of the spacetime \( R^4 \) manifold. Weyl pioneered the use of a Fibre Bundle geometry, first by trying local scale invariance as the generating symmetry of Maxwell's theory, then, realizing
the clash with relativity, happily replacing the $R^1$ group by $U(1)$ and scale invariance by *quantum phase invariance* of the electron’s wave-function, as borrowed from F. London. Weyl himself pointed to the emergence of such a gauge symmetry for the Lorentz group in General Relativity, when he showed that Dirac’s equation required the introduction of Darboux frames (*repères mobiles*), today’s tetrads. Again, nothing like a higher rank *semi-simple Lie algebra* in all this.

Not so with the true *aficionados* of group theory, such as Racah and Wigner. Racah found ways of applying various simple algebras in classifying higher spectra. His methods, later developed and extended by such as L. Biedenharn and M. Moshinsky, exploited higher rank Lie algebras applied to the representation spaces of $SO(3)$. I recall Racah enjoying (anecdotically) the fact that he had found an application for Cartan’s exceptional $G(2)$, in studying the $f$-shell in atomic spectra. One defines an $SO(7)$ algebra acting on some constructs involving the 7-dimensional $f$-shell representation of $SO(3)$ - and the inclusion $G(2) \subset SO(7)$ does it. In these very complicated atomic spectra of the lanthanides, it provides some physical insights. Wigner introduced the *supermultiplet* idea, in which one postulates a basic (nonrelativistic) invariance of the four nucleon isospin/spin states $p^{+1/2}, p^{-1/2}, n^{+1/2}, n^{-1/2}$, constituting the fundamental 4-dimensional representation of $SU(4)$. In 1959, Elliott introduced $SU(3)$ in *nuclear structure*, by postulating 3-dimensional *harmonic oscillator* dynamics, somewhat in the spirit of Pauli’s algebraic treatment of the hydrogen atom from Keplerian dynamics. This $SU(3)$ is reduced over the (maximal) $SO(3)$ subgroup - the angular momentum algebra, saturating the 3-dimensional defining representation. Racah initiated the search for nonlinear operators which could remove the degeneracies, a problem to which Y. Lehrer-Yilamed made important contributions and which was finally solved by Biedenharn. All of this did not involve *exploration* programs and the need to identify a simple Lie algebra of some rank higher than one - or explore its representations.

2. The 1954-64 Search for a Global Symmetry in Particle Physics and the Emergence of the Dynkin Diagrams in Particle Physics

The realization that nuclear binding represents a new and *strong* interaction came in 1932, with the discovery of the neutron: the fact that nuclei are composed of protons and neutrons implied non-electromagnetic forces, also observed to be short ranged. Yukawa suggested a *massive meson* as generating a short range potential, an insight which was finally fully vindicated by the discovery of the pions ($\pi^+, \pi^0$) with a mass of some $130Me/c^2$. Kemmer had indeed shown that the assumption of "$I$-spin" $SU(2)$ invariance required a 3-component $I = 1$ meson multiplet, for an $I = 1/2$ ($N(p, n)$) doublet, coupled via $N(\tau \cdot \pi)N$ ($\tau$ the Pauli matrices, $\tau^a$ an $I$-vector). It turned out, however, that this was not the full story, as new "unneeded" mesons (the $K^0, \bar{K}^0, K^\pm$, around $490Me/c^2$) and new baryons, the $\Lambda^0$ at $1100Me/c^2$, $\Sigma^\pm, \Sigma^0$ around $1200Me/c^2$ and the $\Xi^0, \Xi^-$ at $1300Me/c^2$ (all thus appropriately termed "strange" particles) were found within a short time after the discovery of the pions. The identification of *strangeness* $S$ or *hypercharge* $Y = B + S$ ($B$ the baryon number, the same as the *atomic mass number*), with the relation with *electric charge* $Q$ given by $Q = I^3 + Y/2$ revealed the existence of yet another conservation law (that of $Y$) obeyed by the *strong interactions*. 

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What was the relation between the $3\pi$ and $4K$ mesons? Between the $p, n, \Lambda, 3\Sigma, 2\Xi$?

A search was on, for a larger "global" symmetry, which would incorporate the $SU(2)_I$ of isospin and the $U(1)_Y$ of Yang-Mills. Between 1954 and 1960, models based on $SO(4), SO(7), SO(8), SO(9)$ were suggested by Salam and Polkinghorne, by Schwinger and by Gell-Mann (mathematically clarified by Tiomno), by Salam and Ward. Only rotation groups were tried; physicists were familiar with $SO(3)$ and generalized from that group. It is interesting to note that most of the above researchers and many others had been in Racah's audience at the IAS in Princeton in 1951, when he delivered a lecture series on Lie algebras and Lie groups and their representations. Racah's lectures included a summary of Cartan's 1894 thesis, the classification of the simple Lie algebras, which was all that was needed in order to conduct an organized search. However, whether it was because of Racah's heavy Italian accent (he was self-taught and pronounced English words phonetically as in Italian: "these" sounded like "shay-say" Y) or due to the lack of mathematical preparation of the members of the audience, in conversations with me many years later, they all claimed to have had no notion in 1954-1960 that the lectures they had heard in 1951 were highly relevant to their search effort.

I have related elsewhere how I happened to find myself in May 1960 at a desk at Imperial College in London, aged 35 and after a career in the Israel Defense Forces, about to start a research project (in the framework of a Ph D program) under the guidance of Abdus Salam. He suggested I should study the issue of the acquisition of mass by a gauge boson (geometrically, a connection on a fibre bundle). I was interested in the search for the global symmetry and insisted. Salam gave in, but warned "You are embarking on a highly speculative search; however, if you are going to do it, do it right. Do not be satisfied with the little group theory I could and have taught you. Learn it in depth. Incidentally, I have just heard from Dynkemin that a Russian mathematician named Dynkin (he looked it up in his notes and found a scribbled reference) - see in recent issues of the Transactions of the American Mathematical Society - he classified all subgroups, finding even some which Racah has missed. You could start there!". I was learning that Racah, whom I knew in Israel in 1952-54, when I was representing the Defense Ministry on the Board of the Atomic Energy Commission (still before I knew anything about physics, though with the background of a mechanical and electrical engineer) could have advised me in my present search. However, I set on finding Dynkin's work.

The library's Transactions of the AMI contained no reference to Dynkin. After a few days, however, it occured to me that Dynkin being Russian, the place to look for his work was perhaps in some Translations of the AMS, if there were any. Indeed, the Translations existed; moreover, the catalogue showed two articles by E.B. Dynkin in the 1957 volume, except that the volume itself was not available. I reported to Salam, adding that I was going to use my former "diplomatic" connections to purchase a copy in the USA and have it air-mailed to me urgently. Salam said "order two". I called up my former colleague, Israel's Defense Attaché in Washington and set him on the errand. The books arrived in July and I started reading Dynkin.

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1 I learned in 1978, when I first met Dynkin in the flesh, in my office in Tel-Aviv, that this work had been his doctoral thesis, for which he had been nominated to receive a prestigious prize of the Soviet Academy of Sciences. However, Pontrjagin, who presided over the prize committee, refused to endorse the award for the results which consisted, in his opinion mostly of useless long tables.
After some attempts, I realized what my problem was. These were advanced searches for subalgebras of the simple Lie algebras - but I had no notion what the Lie algebras themselves were like! An appendix in one of the papers referred to a previous work of Dynkin's which apparently contained rules for a diagrammatic methodology, now applied to the subgroups. I looked for that earlier work.

The British Museum Library possessed a heliographed copy, white on blue, the technique used in the Forties and Fifties to copy printed material, exposing the emulsion to sunlight and developing it with ammonia. This turned out to be extremely well written, with elegant mathematical proofs. Dynkin was reproducing Cartan's classification, but in a geometrical version. The root diagrams of the Simple Lie Algebras were shown to allow, in the Euclidean geometry of the r-dimensional Cartan subalgebra (r the rank), for just four values of the angles (90, 120, 135, 150 degrees). Dynkin managed to abstract the entire information about the algebra (its generators and structure constants) and about the representations, by providing a diagrammatic code for the construction of a basis (the simple roots) in the Cartan subspace.

I realized what I was after: an r = 2 Lie algebra (to take care of $I_4^*$, $Y^*$) and containing SU(2)$_f$. I constructed from the diagrams the root diagrams of the algebras $A_2$, $B_2$, $G_2$, $D_2$, $G_2$ (groups, when unitary, SU(3), SO(5), USp(14), SO(14), G(2)). $G_2$ was most intriguing, and moreover, turned out to have a six-pointed star-of-David as root diagram! However, I became very quickly convinced that $A_2$ was "it". Moreover, my reading of the experimental situation was that the six known $J = 1/2^+$ ($J^P$ - spin and relative parity) baryons $N, \Lambda, \Sigma$, and most probably the two $\Xi$ (whose spin had not yet been finalized, the possibilities being $J = 1/2, 3/2$) had to appear in one unitary irreducible representation ("unirep") of the group, here the adjoint, 8-dimensional. After October 1960, I heard from Salam (when I showed him my model, upon his return from his summer travels) that the Nagoya University group, headed by Sakata, had applied the same algebra and group but with the Sakata "triplet" $N, \Lambda$ as the (defining) fundamental representation. This meant that they were relegating the three $\Sigma$ to another ("composite") unirep. The two $\Xi$ were assumed to have $J = 3/2$ and assigned to a higher unirep, together with the Fermi "3/3" resonance (four baryons $\Delta$ with $I = 3/2, J = 3/2, m = 1235 MeV$). The Nagoya group's decisions were accompanied by comments of "philosophers"9, preaching diacritical materialism and stressing the dogmatic strength of the Sakata assignments, based on the belief that there had to be a materialistic foundation, little hard "fundamental" bricks, and these could only be $p, n, \Lambda$, with the right charges needed to construct all states. This was in contrast with my assignments, which were based on a reading of the experimental results, as faithful as possible, and making the best selection in the abstract, leaving the structural aspects to the next stage, rather than making the assignments on the strength of a structural guess, especially when knowing so little about the relevant dynamics. The mesons, on the other hand, could be (and were) assigned to the same octets in both SU(3) models, thus predicting the existence of an eighth 0~, with the seven $\pi, K$ states. Gell-Mann10, who had meanwhile come out with the same suggestions as mine, had, in addition, worked out a mass formula which predicted the eighth meson's mass. By mid 1961, it had indeed been discovered by A. Pevsner and colleagues, at the predicted mass. We were also predicting a $1^- 8 \oplus 1$ of SU(3) from considerations relating to local gauging, convincingly preached.
earlier by J.J.J. Sakurai, with respect to the $SU(2)'\otimes U(1)'$. These predicted nine mesons were all found in 1961-62.

Throughout 1961-63, the Sakata model appeared to many as the most plausible. Dynkin diagrams became popular and three groups carried out similar surveys\textsuperscript{31-33}. Gell-Mann and Glashow proved that a Yang-Mills gauge theory (a fibre bundle with spacetime as base manifold and a Lie group as fibre) required a compact group. P. Ioniades (who had followed me into Dynkin diagrams at Imperial College) proved the same theorem. Meanwhile, the spin of the $\Xi$ was measured and fitted my version of $SU(3)$, but this was not yet universally accepted as a final verdict. In 1962-63, Behrends and Sirlin\textsuperscript{14} proposed a $G_2$ model. The simplest construction, of course, was based on the Dynkin diagram - coding that six-point star weight diagram.

It was only after the prediction\textsuperscript{15} and experimental validation\textsuperscript{16} of the existence of the $\Omega^-$ : $J = 3/2^+$, $m = 1670 \text{ GeV}/c^2$ that the situation was settled, the octet version of $SU(3)$ being universally adopted. Structure soon followed, namely the quark idea\textsuperscript{17-19}.

The successes of $SU(3)$ - selection rules, branching ratios, mass formulae, electromagnetic features (mass differences, anomalous magnetic moments etc.), weak-interactions branchings, etc. - and of its chiral extension $[SU(3)\otimes SU(3)]_{\gamma_1}$, turned group theory into one of the main pillars of particle physics. Textbooks after 1964 carry Dynkin diagrams, as a concentrated way of conveying the key algebraic notions\textsuperscript{20} for the classical groups, and as the only practical way of constructing anything relating to the exceptional.

Throughout 1965-70, the main interest in Particle Physics\textsuperscript{20} centered on spin (or Lorentz) extensions of the unitary spin $SU(3)$, involving the envelopes $SU(6), SL(6, C), U(6)\otimes U(6), U(6, 6)$ of the tensor-multiplications of the spacetime groups $SU(2), SL(2, C), SU(2)\otimes SU(2), U(2, 2) = SO(4, 2) = \text{conformal}[M_{1,3}]$ into unitary spin $SU(3)$. Another interest applied to the hadrons' excitation bands ("Regge sequences") and involved $SL(3, R) \subset SL(4, R)$ of covariance - as $\text{Diff}(4, R)$ in world tensors is represented nonlinearly over its linear subgroup $SL(4, R)$; in curved space, the Hilbert space is given by the Affine $A(4, R)$ or $SA(4, R)$, with the stability subgroup mostly $SL(3, R)$. Another relevant development was the discovery of the double-covering of these groups (world spinors\textsuperscript{21}). Diagrams came in handy but were not essential. And yet, within a few years, the new research directions in Particle Physics generated new algebraic requirements, leading to at least six other applications of Dynkin's algebraic techniques!

3. Applications in the Search for GUTs, in Supergravity and in Superstrings

The years 1971-74 in the Physics of Particles and Fields were those of the emergence of the Standard Model, describing all physical phenomena up to center-of-mass energies of 17 TeV. This is based on a fibre bundle, with spacetime as the base-manifold and with $SU(3)_{\text{color}}\otimes [SU(2)'\times U(1)']$ as structure group (* denotes strong and w electro-weak interactions). In itself, it did not involve new riddles in group or representation theory; however, one direct result was the emergence of a new search, the search for a GUT (gauge unified theory). The fact that the charges of protons and electrons are precisely of the same magnitude is one piece of evidence for the existence of an algebraically irreducible constraint, involving both hadrons and leptons - and there are several additional such proofs. The
success of the semi-unification of the “weak” and the electromagnetic currents in the above (reducible) $^r$-marked group encouraged the search for a simple Lie group containing both the $^s$-marked and $^w$-marked groups. The classification of the simple Lie algebras was again the relevant catalogue - and as a second step, the study of the representations of the candidates - all of this again involving Dynkin’s method. The two main candidates turned out to involve $A_4$ ($SU(5)$) and $E(6)$.

The simplest $SU(5)$ model was experimentally excluded in the eighties (no proton decay observed, bringing the lower bound for the proton lifetime up to $10^{31}$ years). $E(6)$ is still open!

The second avenue was opened by the development of Lie superalgebras and supergroups $^{23}$. V. Kac classified the simple Lie superalgebras$^{24}$. There arose an interest in the representations - and the Dynkin diagrammatic methods were extended to cover superalgebras$^{25}$. The most important application of superalgebras in Particle Physics has consisted in a superalgebraic extension of the Poincaré algebra, relating particles with Bose-Einstein quantum statistics to particles with the opposite (Fermi-Dirac) statistics. This is known as supersymmetry and appears essential to our understanding of the stability of GUTs and of the Standard Model itself$^{26}$. Experimental proof is expected around 2005, when the “Large Hadron Collider” accelerator will be active at CERN. The number $N$ of odd generators in supersymmetry is optional, $N$ “square roots” of the spacetime translations, themselves spinorial anticommutative translations in superspace, the group quotient supersymmetry/Poincaré. In 1976, we concluded, M. Gell-Mann and myself$^{27}$, that the largest allowed case would be $N = 8$, due to theorems forbidding the existence of massless particles with spins larger than $J = 2$. The $N = 8$ case appeared especially appropriate for a supersymmetrization of the graviton ($J = 2$) and for a unification of gravity with all other interactions.

Applying supersymmetry to Einstein’s gravity had indeed just yielded Supergravity with $N = 1$ $^{28}$. One by one, the higher $N$ theories were now being constructed, until Cremmer and Julia produced the $N = 8$ theory$^{29}$, in a construction involving gravity in an 11-dimensional spacetime. The $N = 1$ supergravity in 11 dimensions reduces to an $N = 8$ supergravity in our 4 dimensions. The remaining 7 dimensions become (spontaneously) compactified$^{30}$ and yield internal symmetries (like our $SU(3)$). Miraculously, the maximal symmetry of these 7 dimensions$^{30}$ is a noncompact real form of $E(7)$! The subject is alluring, because the reductions of this $E(7)$ exploit the wealth of Milnor’s 7-sphere, octonions etc., but in any case, if this version of supergravity is indeed selected by nature, there is a lot that can be done with Dynkin’s methods.

One surprising result$^{31}$ is that if we require spacetime to have only 3 dimensions, the maximal internal symmetry becomes $E(8)$. If, on the contrary, we reserve $d = 5$ out of the 11 dimensions for the residual spacetime $R^d$, we get $E(6)$ for the internal symmetry; the internal symmetry is thus $E(11 - d_R)$. But Cartan’s classification has no $E_r$ for $r < 6$. Yet the geometry requires their existence - and they indeed come out, as we might guess from the Dynkin diagrams, by continuing to cut out one little circle each time, along the horizontal line, i.e. $E_6 = D_5$, $E_4 = A_4$, $E_3 = A_2 \oplus A_1$, $E_2 = D_2$, $E_1 = A_1$.

The next developments relate to a hypothetical theory going beyond GUTs and their supersymmetrization, the theory of the Quantum Superstring$^{32}$ (“QSS”). This supposed “TOE” (“theory of everything”) contains quantum gravity, together with...
all other interactions. Truncating it underneath Planck energies \(10^{19}\text{GeV}\), or for lengths above Planck lengths \(10^{-33}\text{cm}\), beyond which gravity becomes quantal and strong and where spacetime itself is also quantized, we should get a supersymmetrized GUT (i.e. a Relativistic Local Quantum Field Theory), extended to include the generations structure (the entire system of quarks and leptons appears in three "copies"). It was found in 1984 that the QSS avoids some bad pitfalls of RLQFT - infinities known as the "anomalies" - provided their symmetry algebras are either \(E_8 \otimes E_8\) or \(D_{16}\). These anomalies were the only divergences expected in QSS, so that their cancellation was greeted with great enthusiasm. \(E_8 \otimes E_8\) appears to fit much better, as it contains the SM group, etc. Once again, the field was dominated by the exceptional, in an extensive way, and Dynkin diagrams were again essential. Moreover, the QSS exists in \(D = 10\) dimensions and the application of reduction techniques (down to \(d = 4\)) involves Kac-Moody (Affine) Lie algebras\(^{33}\). To treat these new algebras, the technique of Dynkin diagrams has been further developed. The Affine Kac-Moody algebras have extended Dynkin diagrams\(^{34}\), which are again essential to their understanding and application. Moreover, the specific construction involves one type which is indeed distinguished by its Dynkin diagram - the "single-laced" ADE.

The most recent boost is related to a model known as "M theory"\(^{35}\). It has been found that the different versions of QSS theory, which were thought to be unrelated, are tied together by a mathematical transformation, "duality". This is an overlap of Hodge duality (going from \(n\)-forms to \(D - n\)-forms) with the "electric-magnetic" duality of Maxwell's equations, with the "strong-weak" duality implied in Dirac's monopole quantization \(eq = N\) and with the duality of the bootstrap-generated dual models: particles B are created as soliton solutions of equations in which the fundamental objects are particles A - and vice versa, the A appear as solitonic solutions in a theory in which the B are fundamental\(^{36}\). This requires an extension of the QSS system to \(D = 10 \oplus 1 = 11\), but the prize is that the \(d = 2\) membrane in \(D = 11\), which in the reduction back to \(11 = 10 + 1\) becomes \(2 = 1 \oplus 1\) (the string), also contains as truncation the \(D = 11\) \((N = 8)\) supergravity with its beautiful \(E_7!\) How much more joy with Dynkin diagrams!

4. Dedication

I am happy to repay a debt to E.B. Dynkin, incurred 39 years ago. I wish him many more fruitful years of original, imaginative and useful mathematics!
References


