In the plane the shortest distance between two points is the length of a straight line segment connecting those two points. In the sphere the (shortest) distance between two points \( p_1 \) and \( p_2 \) is \( d(p_1, p_2) \), the shorter of the lengths of the two geodesic arcs connecting \( p_1 \) and \( p_2 \).

1. Show that the function \( d \) defined above is a distance function. Namely, for three points \( p_1, p_2, p_3 \in S^2 \), \( d(p_1, p_2) + d(p_2, p_3) \geq d(p_1, p_3) \). (Hint: Use the formula for the geodesic length in terms of the chordal length, recall that \( \sin^{-1} \) is a convex function, and the triangle inequality holds for Euclidean distances.)

2. In the unit sphere \( S^2 \) let \( p_1, p_2 \) be two points in the northern hemisphere such that the north pole \( N \) is the midpoint on the shorter geodesic arc between them. Let \( p_3 \) be any point on the equator \( E \) that is equidistant between \( N \) and the south pole \( -N \). Show that

\[
d(p_1, p_3) + d(p_3, p_2) = \pi.
\]

(Hint: Some reflection is required.)

3. (a) Suppose that a simple closed curve \( C \) is of length less than \( 2\pi \) in \( S^2 \). Show that it is contained in a hemisphere. (Hint: Look at the midpoint of the shortest geodesic between two points on \( C \) that divide it into two equal length arcs, and apply Problem 2 and Problem 1.)

(b) Show that a simple closed curve \( C \) of length less than \( 2\pi \) in \( S^2 \) bounds two regions, one whose area is strictly less than \( 2\pi \) and another whose area is strictly greater than \( 2\pi \).

(c) Let \( C \) be an oriented simple closed curve in \( S^2 \). It bounds two regions one to the right as you proceed along \( C \) and one to the left. If \( C \) is deformed continuously, the areas of those two regions vary continuously as well. If length of \( C \) is fixed at a number less that \( 2\pi \), and the right hand area starts being less than \( 2\pi \), the right hand area never becomes greater than \( 2\pi \).

4. Suppose that you try to turn a paper bag inside out continuously, but you only use a finite number of folds at a corner. So as the motion continues, a little sphere, with a fixed radius, centered at the corner intersects the paper in a simple closed curve with fixed length. With that restriction, can you turn the paper bag inside out? Explain why or why not.