Chapter 4

Drawing a square and inverse problems
Math 4520, Spring 2015

4.1 Artistic definitions

Let us return to our artist painting some object in the picture plane. There is more to the picture than incidence properties such as Desargues’ Theorem. In fact, if the artist draws accurately what is seen, things such as Desargues’ Theorem will take care of themselves. On the other hand, the picture is a very distorted representation of the object. How distorted can it be? What are some properties that we can measure in the picture that will give us information about the object and how it was painted? We make a few definitions that every artist uses whether he knows it or not.

The artist’s eye, or center of projection, is called the *station point*. All the points in the object plane are projected along a straight line through the station point into the picture plane. The line from the station point perpendicular to the picture plane is called the *line of sight*. The unique point on the line of light in the picture plane is called the *center of vision*. Note that the center of vision is the nearest point on the picture plane to the station point. Normally one tries to keep the center of vision near the center of the actual physical painting. The distance between the station point and the center of vision is called the *viewing distance*. See Figure 4.1

It is usually desirable to keep the viewing distance within some reasonable range. For me 14 inches seems about right. It is not necessary to be ultra exact about these quantities, but if they are grossly out of line, the picture will probably look “wrong”. On the other hand, an artist might deliberately draw with a very short viewing distance in order to enhance the three-dimensional effect. For example, later we will see that when the station point is at infinity, a cube will have two possibilities as to which face is in front. This reversal effect can be disturbing if it is unintended.

Recall that the points at infinity in the object plane can be projected onto ordinary points (sometimes called *finite points*) in the picture plane. We call the projections of these ideal points, the *horizon* in the picture plane. Note that if the object plane is perpendicular to the picture plane, then the center of vision lies on the horizon.

If two lines are parallel in the object plane, then they project onto lines that intersect on the horizon in the object plane. This point of intersection will be called the *vanishing point*.
for the parallel lines. In fact, any single line (not the line at infinity) has an ideal point on it, on the line at infinity. The projection of this ideal point will be regarded as the vanishing point of that single line. Explicitly, the line through the station point parallel to the given line (in the object plane say) intersects the object plane at the vanishing point.

### 4.2 One point perspective

What happens if one of the sides of a square is parallel to the intersection of the picture plane and the object plane? This is called one point perspective. For example, the projection of a square grid in one point perspective is shown in Figure 4.2. The center of vision is then the vanishing point of the side of the square not parallel to the horizon line. The viewing distance is then the distance from the center of vision to either of the $45^\circ$ diagonal vanishing points.

![Figure 4.1](image1.png)

![Figure 4.2](image2.png)

When both vanishing points of the sides of the projection of the square are finite points, then we say that the square (or the square grid it generates) is in two point perspective. For
instance, the skew square in the grid of Figure 4.2 is shown generating its own grid in two point perspective in Figure 4.3.
4.3 Projecting a square and locating the center of vision

Suppose we have a square in the object plane, not necessarily in one point perspective. We will describe a construction in this object plane that will help us locate the center of vision in the picture. Imagine we are looking down on the whole picture drawing process from a long distance away, a bird’s eye view. Then the picture plane is viewed edge-on and we see the square with no distortion. We assume that the picture plane and the object plane are perpendicular. (This means that the lines in both planes, perpendicular to the line of intersection, are perpendicular to each other.) Let $L$ be the line of intersection of the object plane and the picture plane. Figure 4.4 and Figure 4.5 show a construction for a line perpendicular to the horizon line.

Start with the given square.

Draw Line 1 parallel to $L$ through one corner of the square.

Draw Line 2 parallel to the diagonal of the square starting at the other end of Line 1.

Draw Line 3 parallel to one side of the square starting at the other end of Line 2.

Figure 4.4

It is easy to check (use the similar triangles indicated in Figure 4.5, as well the corresponding equal angles) that Line 4 is perpendicular to Line $L$.

Note that the construction never uses the ability to draw perpendicular lines explicitly. Line 4 turns out to be perpendicular to Line $L$, but its construction uses only the ability to draw parallel lines.

We now mimic this construction in the picture plane as shown in Figure 4.6 and 4.7. We simply remember that parallel lines in the object plane project to lines in the picture plane that intersect at their vanishing point at infinity. All the vanishing points are on the horizon line, which is parallel to the Line $L$.

Since Line 4 is perpendicular to the line $L$ in the object plane, the vanishing point of the corresponding Line 4 on the horizon in the picture plane will be the center of vision. This is because Line 4 in the object plane is parallel to the Line from the station point to the center of vision, perpendicular to the picture plane. See Figure 4.7.
Thus we now have a simple construction that allows us to locate the center of vision, assuming that we have a picture of a square in a perpendicular object plane.
4.4 Determining the viewing distance

We now take up the problem of finding how far away from the picture plane we should stand to view a square properly. In other words, we want to find the viewing distance. We make the same assumptions as in Section 4.3. The object plane is perpendicular to the picture plane and we have a square in the object plane projected into the picture plane, as seen in Figure 4.8.

Here again we project the whole configuration into the object plane so that the picture plane is projected onto a line. It is viewed edge-on. In Figure 4.8 we show this parallel projection perpendicular to the object plane. (This is projection from the point at infinity on the line perpendicular to the object plane.)

From Figure 4.8 it is clear that if one draws the lines from the station point parallel to the adjacent sides of the square in the object plane, then these two lines are perpendicular in Euclidean 3-space and well as in Figure 5.2. So the station point and the two vanishing points form a right triangle with the line of sight as the altitude. The viewing distance is the length of this altitude. Note that the vanishing points and the station point lie on a plane parallel to the object plane.

So if one wishes to find the proper point to view a square, one can combine the bottom half of Figure 4.8 and the construction above to construct the center of vision and the “folded down” flap that determines the viewing distance. Since the triangle in 4.8 is a right triangle, it lies on a circle whose center is the midpoint between the two vanishing points of the sides of the projected square.

If one wishes a more analytic description of the viewing distance, let $d_1$ and $d_2$ be the distances from the center of vision to the two vanishing points for the two sides of the projected square. Then the viewing distance $d$ is the geometric mean of $d_1$ and $d_2$. (This is a good exercise in Euclidean geometry.) In other words,

$$d = \sqrt{d_1 d_2}.$$
4.5 Determining the square

Notice that the vanishing point for the 45° diagonal is between the two vanishing points for the two sides. Also the line from the 45° vanishing point to the corner of the square above the horizon line in Figure 4.8, when folded out actually makes an angle of 45°. This fact can be used to run the construction another way. Suppose that the vanishing points for the sides of the square are known (or chosen), and the center of vision is known. Then the corner of the right triangle above the horizon lies on the circle as before. This corner also lies on the
line through the center of vision perpendicular to the horizon line. The intersection of this line and the semi-circle determines the corner. Then the right angle can be bisected. Where this bisecting line intersects the horizon line will be the vanishing point for the $45^\circ$ diagonal of the projection of the square. Once all three vanishing points for the square are known (or constructed), then the projection of the square can be readily drawn. See Figure 4.9, where the order of construction is indicated. Notice that this construction uses more than just the incidence structure in that at a crucial point we need to bisect an angle. Implicitly a compass is used.

![Figure 4.9](image)

### 4.6 Exercises

In problem 1, you will need to copy some pictures that have good examples of one or two point perspective onto a transparency. (Be careful to get the correct sort of transparency for this. The wrong type will melt in the copying machine.) You can use some of the sketches in Geometer’s Sketchpad (or you can make your own, you can use pencil and paper, or use whatever software you are comfortable with) to locate the center of vision and the viewing distance.

1. Using Geometer’s Sketchpad (or other methods, if you like) to locate the center of vision and the viewing distance for your original square and the drawings in the handouts and other pictures that you can find. Try to find 4 or 5 drawings for this exercise. Hand in a copy of the transparency and the final display as you have positioned the points for the calculation.

2. Suppose that a standard $8\frac{1}{2}''$ by $11''$ piece of paper has a picture of a square with the vanishing points of both sides on the paper. If the paper is held as this paper is read, long side up, what is the largest the viewing distance can be? Is this a comfortable distance for you?

3. Suppose one has a quadrilateral in the picture plane below the horizon with all internal angles less than $180^\circ$. Using the construction outlined in Sections 4.3 and 4.4, one finds a station point in 3-space. When the quadrilateral is projected back into object plane, do we get a square? In other words, do all reasonable quadrilaterals in the picture plane come from a square in the object plane? (Hint: Use the $45^\circ$ diagonal line.)
4. Prove the statement claimed earlier that the altitude of a right triangle is the geometric mean of the two segments of the hypotenuse.

![Figure 4.10](image)

5. Justify the claim in Section 4.2 about the viewing distance for one point perspective.

6. Figure 4.11 shows a picture of Mr. Stickler (name borrowed from Richter-Gehbert) in part of a grid in the object plane. Draw a picture of Mr. Stickler in his grid in one-point perspective, but be sure to show what the picture is when the object plane is continued behind the artist’s eye and Mr. Stickler is projected above the horizon line.

![Figure 4.11](image)