Chapter 3

Solid geometry and Desargues’ Theorem
Math 4520, Spring 2015

3.1 The extended Euclidean 3-space

We can regard the Euclidean plane as defined as the set of ordered pairs of real numbers. In this approach the coordinate system becomes part of the definition. Similarly, Euclidean space can be regarded as the set of ordered triples of real numbers. Then it is easy to define points, lines, and planes as solutions to linear equations.

The properties of the incidence relations become a bit more complicated. A point may lie in (be incident to) a line or a plane, but a line and a plane may intersect in a point, or the line may be completely contained in plane. There are many other cases to consider as well. In any case we can add a plane of points and lines at infinity to get the extended Euclidean 3-space, just as we did to the Euclidean plane to get the extended Euclidean plane. Here are the elements (points, lines, and planes) of extended Euclidean 3-space other than the ordinary points of Euclidean 3-space.

A point at infinity is identified with an equivalence class of parallel lines in Euclidean 3-space.

A line at infinity is identified with an equivalence class of parallel planes in Euclidean 3-space.

The plane at infinity is one additional plane that is incident to all the points and lines at infinity.

It will be an exercise to provide the appropriate definitions telling when two elements are incident. This is a little like playing God. Try not to abuse the privilege. It will also be left as an exercise to create a reasonable set of axioms for projective 3-space. Note that the extended Euclidean 3-space that you finish defining must at least satisfy your set of axioms. Indeed we want to have some basic properties to be able to work in projective 3-space. For example we need to be able to define “projection”. The following is some motivation for this idea.

Suppose that an artist wants to draw some object, even an object in a plane. How is this done accurately? Imagine the artist’s canvas as a transparent plane with the object on the other side from the artist’s eye. Call the artist’s eye (we ignore binocular vision) the station point or the center of projection. If the object to be drawn lies in a plane itself, we call that
plane the object plane. See Figure 3.1. For every point \( p \) in the object plane consider the line through (incident to) it and the station point. This line will intersect the picture plane at a unique point \( q \). (\( q \) will be the unique point incident to the picture plane and the line.) We say that \( q \) is the projection of \( p \), (from the station point). We regard \( q \) as a function of \( p \) and indeed we write \( q = f(p) \). So \( f \) is a function whose domain is the set of points incident to the object plane and whose range is the set of points incident to the picture plane. We call \( f \) a projection from the object plane to the picture plane.

Notice that our definition of a projection is purely formal. All we need is two distinct planes, one we can call the object plane and the other we can call the picture plane. Of course we must also be careful to choose the center of projection so that it is not incident to either of these planes. In the extended Euclidean 3-space, as well as your generally defined projective 3-space, any line through (incident to) the center of projection must intersect each of the the planes in a unique point. (The line must be incident to a unique point which is also incident to each of the two planes.) So projection is well-defined.

Here are some simple observations:

1. The projection of a line is a line. (The projection of the points incident to a line in the object plane is the set of points incident to a line in the picture plane.) Hence projection can be equivalently regarded as a function taking the lines of the object plane to the lines of the picture plane.

2. Projection is a one-to-one and onto function.

3. In your extended Euclidean 3-space model, the projection of the line at infinity of the object plane can be an ordinary line in the picture plane. You can draw a picture of the horizon.

The creation of the projective plane is great, but what can you do with it? Can you prove anything? This is a silly question for a mathematician, who can prove something about anything. But honestly it is difficult to prove something interesting using just the bare bones three axioms of the projective plane. However, if one uses projective 3-space, then it is possible to prove some remarkable statements that are intimate with the whole foundation of Geometry.

It is useful to have one other little Lemma with regard to Axiom 3 for the projective plane.
Lemma 3.1.1. In a projective plane (or our projective space), every line is incident to at least three distinct points (or dually every point is incident to at least three lines).

Proof. Recall that Axiom 3 holds and let \( L \) be any line. Then there are four distinct points \( p_0, p_1, p_2, p_3 \) in the projective plane, no three collinear. Without loss of generality we can assume that \( p_0 \) is not incident to \( L \). Then the projection from \( p_0 \) of \( p_1, p_2, p_3 \) into \( L \) are the three required points. See Figure 3.3.

Figure 3.3

To properly state one very basic result, we start with a few definitions. We say that the ordered triple of points \( [p_1, p_2, p_3] \) is a point triangle if the points are not incident to any line. Similarly, and ordered triple of lines \( [L_1, L_2, L_3] \) is a line triangle if the three lines are not incident to a point. Note that to each point triangle \( [p_1, p_2, p_3] \) we can associate a line triangle \( [L_1, L_2, L_3] \), where \( L_3 \) is the unique line incident to \( p_1 \) and \( p_2 \), \( L_2 \) is the unique line incident to \( p_1 \) and \( p_3 \), etc. See Figure 3.4. Often we will simply say “triangle” where we will mean one of these two definitions.

Suppose \( [p_1, p_2, p_3] = \Delta_1 \) and \( [q_1, q_2, q_3] = \Delta_2 \) are two point triangles. We say that \( \Delta_1 \) and \( \Delta_2 \) are in perspective with respect to a point \( p \) (and \( p \) is the point of perspectivity) if the lines determined by corresponding points are all incident to \( p \). In other words, \( p, p_i, \) and \( q_i \) are collinear (all incident to a line), for \( i = 1, 2, 3 \). See Figure 3.5. Similarly, we say that two (line) triangles are in perspective with respect to a line \( L \), if the three points incident to corresponding lines all are incident to \( L \).

Note that corresponding elements in these definitions (for example \( p_1 \) and \( q_1 \)) must be distinct in order for there to be a unique element (a line for example) determined by them. We can now state a Theorem due to Desargues, who was an architect at Lyon, France in the seventeenth century.
Theorem 3.1.2 (Desargues). Two triangles are in perspective with respect to a point if and only if they are in perspective with respect to a line.

Desargues of course only meant his Theorem to apply to the extended Euclidean plane. But the statement makes sense in any projective plane since it is only concerned with incidence relations. In general, the above statement is not true for all projective planes. However, we shall give a proof below that applies when the projective plane is part of a projective 3-space. This is certainly true of the extended Euclidean plane. Note also that we need only to prove the “only if” part of the statement, since if we reverse the words “point” and “line” and apply the same proof that will prove the “if” part of the statement.

Proof of Desargues’ Theorem. We shall first generalize the statement of Desargues’ Theorem to include the cases when the two triangles are in a projective 3-space, not just necessarily in a projective plane. You should check that the definitions of being in perspective with respect to a point or to a line apply equally well in the projective 3-space.

Start with the three coordinate axes.
Consider two point triangles, where each triangle has one vertex on each axis as in 3.7. These two shaded triangles are in perspective with respect to a point (the origin in our case).

Since the four points of the two triangles all lie on the y and z axes, they all lie in the shaded plane, the $yz$–plane as in Figure 3.8. Thus the corresponding sides (when extended in this picture) are incident to a point in the $yz$–plane (a black dot in Figure 3.8).
Similarly for the other two sides (when extended), we obtain two more points as in Figure 3.9.

But each of these points (the black dots) is incident to one (extended) side of each triangle. So they are all incident to both planes of the two triangles we started with. Hence these three points are incident to the line of intersection (incident to) the planes of those triangles as in Figure 3.10.

So the two triangles are in perspective with respect to a line, which finishes Desargues’ Theorem. . . . Or does it? We neglected to consider the case when the two triangles (or more precisely the points and lines of these two triangles) are both incident to the same plane, which is precisely the case we started with. But we can save things if we can project the points and lines in projective 3-space in such a picture as above into any given configuration (a collection of points and lines) in the projective plane. We do that as follows.

Start with the two triangles that are incident to the same plane as in 3.11. We assume that these two triangles are also in perspective with respect to a point in that plane.

Choose a new point not incident to this plane. Regard this new point as a point of
projection from projective 3-space into the given plane as in Figure 3.12.

Choose another point (in black in Figure 3.13) incident to the line incident to the point of projection and the point of perspectivity in the plane. This new point will be the point of perspectivity in projective 3-space, and it must be chosen not equal to the point of projection or the point of perspectivity in the plane. This can be done by using Lemma 3.1.1. Next choose the point indicated in Figure 3.13, that will be the corresponding point of the new triangle.

Do the same for all three pairs of corresponding points as in Figure 3.14.

Now we can really project the configuration in projective 3-space into the plane of one of the triangles, and we have shown that the two triangles are in perspective with respect to same line of perspectivity. This finishes our “proof” of Desargues’ Theorem. □
Since the four points of the two triangles all lie on the y and z axes. Hence these three points are incident to the line of intersection (incident to) the planes of the shaded triangles.

Figure 3.10

Figure 3.11

Figure 3.12
Do the same for all three pairs of corresponding points as below:

- Point of projection
- Corresponding point of the new triangle
- Point of perspectivity in 3-space
- Point of perspectivity in the plane

No, we can project the configuration in projective 3-space into the plane of one of the triangles, and we have shown that the two triangles are in perspective with respect to the same line of perspectivity. This finishes our proof of Desargues' Theorem.
3.2 Exercises:

1. Complete the definition of the extended Euclidean 3-space.

2. Finish the definition of projective 3-space so that the above proof of Desargues’ Theorem is valid. You may only include axioms that are valid for the extended Euclidean 3-space. Your axioms should also have the property that the words “point” and “plane” can be interchanged while the word “line” stays the same.

3. In the definition of being in perspective with respect to a point we note that corresponding points must be distinct. What happens if they are not distinct? Is Desargues’ Theorem still true? What happens if corresponding lines are not distinct?

4. In the Fano projective plane of order 2, does Desargues’ Theorem hold? Try a few cases. But be careful to allow yourself to consider cases when some of the six points of the two triangles coincide. Some coincidences are forbidden by the hypothesis of Desargues’ Theorem and some are not.

5. The notion of two triangles being in perspective with respect to a point depends on the correspondence between the points of the two triangles. Can two triangles be in perspective with respect to different correspondences?

6. (Tricky) It was noticed that there was a duality in 3-space where “points” and “planes” are interchanged. In the proof of Desargues’ Theorem the notion of two (point) triangles in 3-space being in perspective with respect to a point was generalized from the plane. A triple of planes can be viewed as a polyhedral “corner”. Draw a picture of this and show how the statement of Desargues’ Theorem holds in this context.

7. In Chapter 2 we discussed projection from a point as a function from the points incident to one line to the points incident to another line. What is the analogous function for the projection from a line in the plane? What about projection from a plane in 3-space?

8. It was mentioned that projections are one-to-one and onto. Where do points above the horizon in the picture plane come from (under f) in the object plane? (Use the formal definition of f. Our artist has an eye in the back of his head.)

9. In the following Figures 3.15 and 3.16 there are two shaded triangles. In each case record the center of perspectivity and the correspondence between the vertices of the two triangles. Then describe the line of perspectivity for the two pairs of triangles. Each line can be described by two distinct points that are incident to it, and any point can be described by the letter in the figure or by two lines that are incident to it. For example, the triangle ABC is one of the pair of triangles that are in perspective in both figures.
Figure 3.15

Figure 3.16