1. Give an ordinary generating function for the sequence \( a_n \) defined by \( a_0 = 1, a_1 = 3, \) and \( a_n = a_{n-1} + 2a_{n-2} \) for all \( n \geq 2. \)

We compute

\[
f(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= 1 + 3x + \sum_{n=2}^{\infty} a_n x^n \\
&= 1 + 3x + \sum_{n=2}^{\infty} (a_{n-1} + 2a_{n-2}) x^n \\
&= 1 + 3x + \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} 2a_{n-2} x^n \\
&= 1 + 3x + x \sum_{n=1}^{\infty} a_n x^n + 2x^2 \sum_{n=0}^{\infty} a_n x^n \\
&= 1 + 2x + x \sum_{n=0}^{\infty} a_n x^n + 2x^2 \sum_{n=0}^{\infty} a_n x^n \\
&= 1 + 2x + xf(x) + 2x^2 f(x) \\
(1 - x - 2x^2) f(x) &= 1 + 2x \\
f(x) &= \frac{1 + 2x}{1 - x - 2x^2}
\]
2. For any positive integer \( n \), show that there is some value of \( c \) (which can depend on \( n \)) such that for all \( k \geq c \), \( p_k(n + k) = p_c(n + c) \). Also find the minimum such value of \( c \).

\( p_k(n + k) \) is the number of ways to put \( n + k \) indistinguishable balls into \( k \) indistinguishable boxes such that no box is empty. If we remove one ball from each box, it is the number of ways to put \( n \) balls into \( k \) boxes, with some boxes allowed to be empty. If \( k > n \), we must leave at least \( k - n \) boxes empty, and because the boxes are indistinguishable, removing these boxes that must be empty does not change the number of arrangements. Therefore, for all \( k \geq n \), \( p_k(n + k) = p_n(2n) \). Furthermore, \( n \) is the minimum possible such value of \( c \), as if if \( c < n \), then \( p_c(n + c) \) does not count the possibility of each extra ball having its own box, so we lose at least one of the arrangements that would be counted in \( p_n(2n) \).
3. How many positive integers are there that are factors of at least one of $2^3 \cdot 3^5 \cdot 7$, $3^5 \cdot 5^9$, and $2^8 \cdot 5^5$?

We use inclusion-exclusion. Any factor of $2^3 \cdot 3^5 \cdot 7$ must be of the form $2^m \cdot 3^n$ for $m \leq 4$ and $n \leq 7$. There are 5 possible choices for $m$ and 8 possible choices for $n$, so there are 40 factors of $2^3 \cdot 3^5 \cdot 7$. Similarly, there are 60 factors of $3^5 \cdot 5^9$ and 54 factors of $2^8 \cdot 5^5$.

Any number that is a factor of both $2^3 \cdot 3^5 \cdot 7$ and $3^5 \cdot 5^9$ must be a factor of their greatest common divisor, which is $3^5$. There are 6 such numbers. Similarly, there are 6 numbers that divide both $3^5 \cdot 5^9$ and $2^8 \cdot 5^5$, and five that divide both $2^3 \cdot 7$ and $2^8 \cdot 5$. Finally, 1 is the unique common divisor of all three numbers. Therefore, by inclusion-exclusion, the number of factors of at least one such number is

$$40 + 60 + 54 - 6 - 6 - 5 + 1 = 138.$$
4. Give a formula for the Stirling number of the second kind $S(n, 2)$.

The Stirling numbers of the second kind are the ways to put distinguishable balls into indistinguishable boxes with no box empty. There are $2^n$ possible ways to place the $n$ balls into two distinguishable boxes. Two of these put all of the balls into the same box, so there are $2^n - 2$ arrangements for distinguishable boxes. Divide by $2!$ ways to rearrange the boxes to get $2^{n-1} - 1$ ways to place the balls into indistinguishable boxes.
5. Give an exponential generating function for the sequence \( a_n \) defined by \( a_0 = a_1 = 1 \) and \( a_{n+1} = a_n + n(n-1)a_{n-1} \) for all \( n \geq 1 \).

We compute

\[
\begin{align*}
  f(x) &= \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \\
  f'(x) &= \sum_{n=1}^{\infty} \frac{a_n x^{n-1}}{(n-1)!} \\
  &= \sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!} \\
  &= \sum_{n=0}^{\infty} (a_n + n(n-1)a_{n-1}) \frac{x^n}{n!} \\
  &= \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} + \sum_{n=0}^{\infty} (n-1)a_{n-1} \frac{x^n}{n!} \\
  &= f(x) + \sum_{n=2}^{\infty} \frac{a_{n-1} x^n}{(n-2)!} \\
  &= f(x) + x^2 \sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!} \\
  &= f(x) + x^2 f'(x).
\end{align*}
\]

This gives us \((1-x^2)f'(x) = f(x)\), or equivalently, \( f'(x) = \frac{1}{1-x^2} f(x) \). If \( f(x) = e^{g(x)} \), then \( f'(x) = g'(x) e^{g(x)} = g'(x) f(x) \), so we have \( g'(x) = \frac{1}{1-x^2} \).

Then

\[
\begin{align*}
  g(x) &= \int g'(x) \, dx \\
  &= \int \frac{1}{1-x^2} \, dx \\
  &= \int \frac{1}{2} \left( \frac{1}{1+x} + \frac{1}{1-x} \right) \, dx \\
  &= \frac{1}{2} \left( \ln(1+x) - \ln(1-x) \right) + C \\
  &= \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) + C \\
  &= \ln \left( \sqrt{\frac{1+x}{1-x}} \right) + C.
\end{align*}
\]

From this, \( f(x) = e^{g(x)} = \sqrt{\frac{1+x}{1-x}} e^C \). If \( x = 0 \), then \( f(0) = a_0 = 1 \). The function gives \( f(0) = \sqrt{\frac{1+0}{1-0}} e^C = e^C \), so \( e^C = 1 \). Therefore, \( f(x) = \sqrt{\frac{1+x}{1-x}} \).
6. Let \( P \) be the set of partitions of 150 such that for all \( k \geq 1 \), if there is a part of size \( k+1 \), then there is at least one part of size \( k \). Show that the number of partitions in \( P \) for which the largest part is even is equal to the number of partitions in \( P \) for which the largest part is odd.

Let \( P^* \) be the set of conjugates of partitions in \( P \). If a partition is in \( P \), then if there is a \((k+1)\)-th column of the Ferrers diagram, then the \( k \)-th column is longer, because there is a part of size \( k \). Therefore, no two columns have the same length. Columns of a partition are rows of the Ferrers diagram in its conjugate partition, so for every partition in \( P^* \), no two rows are the same length. This means that all partitions in \( P^* \) have no two parts of the same size. These steps work in reverse also, so \( P^* \) is the set of partitions of 150 with no two parts equal.

The size of the largest part of a partition is the number of parts of its conjugate partition. Therefore, the problem asks us to show that the number of partitions in \( P^* \) with an odd number of parts is equal to that for an even number of parts. If 150 is not a pentagonal number, then this was a theorem in class, and is essentially Theorem 15.5 in the book. We can check \( \omega(10) = 145 \), \( \omega(11) = 176 \), \( \omega(-10) = 155 \), and \( \omega(-9) = 126 \), so if \( \omega(m) = 150 \), then \( 10 < m < 11 \) if \( m > 0 \) and \( -10 < m < -9 \) if \( m < 0 \), both of which are obviously impossible.