Math 4200, Problem set 4

Solutions

October 3, 2013

Problem 1. $i^{2013}$

Solution. It follows from the fact $i^4 = 1$ that $i^{2013} = i$. □

Problem 2. $\left(\frac{1+i}{1-i}\right)^{2013}$

Solution. After simplification we see that $\frac{1+i}{1-i} = i$, which reduces to Problem 1. □

Problem 3. $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^{2013}$

Solution. Simplify $\frac{1+i\sqrt{3}}{1-i\sqrt{3}}$ to get $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$. Note that this is a cube root of 1, hence $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^{2013} = 1$. □

Problem 4. $z^3 = 8$

Solution. Clearly 2 is a solution hence all solutions are given by: 2, $2\omega$, $2\omega^2$; where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ is a primitive cube root of 1 (note that $\omega^2 = \overline{\omega}$). We get the three solutions:

$$2, \quad -1 + \sqrt{3}, \quad -1 - \sqrt{3}.$$

□

Problem 5. $z^3 = -8$

Solution. We get the solutions of this equation by multiplying by $-1$ the solutions from Problem 4.

$$-2, \quad 1 - \sqrt{3}, \quad 1 + \sqrt{3}.$$  □

Problem 6. $z^6 = 64$

Solution. If $z^6 = 64$ then $z^3 = \pm 8$ so the set of solutions to this equation is the union of the solutions from Problem 4 and the solutions from Problem 5. □
Problem 7. \( z^6 = -64 \)

Solution. Since \( i^6 = -1 \) we can get all solutions to this equation by multiplying by \( i \) the solutions from Problem 6.

Problem 8. \( z^4 = -4 \)

Solution. All solution must be of the form \( \zeta \sqrt{2} \), where \( \zeta^4 = -1 \). Since \(-1 = e^{\pi i} \) we conclude that the fourth roots of \(-1 \) are of the form \( e^{\pi i/4} \). We obtain the following solutions:

\[ 1 + i, \quad -1 + i, \quad -1 - i, \quad 1 - i. \]

Problem 9. \( \ddot{x} - 9\dot{x} + 8x = 0 \)

Solution. The characteristic equation \( \lambda^2 - 9\lambda + 8 \) has 1 and 8 as roots. The general solution is given by

\[ x(t) = \alpha e^t + \beta e^{8t}, \quad \alpha, \beta \in \mathbb{R}. \]

Problem 10. \( x^{(4)} - 9\ddot{x} + 8x = 0 \)

Solution. The characteristic equations factors as \( (\lambda^2 - 1)(\lambda^2 - 8) \). The general solution is given by

\[ x(t) = \alpha_1 e^t + \alpha_2 e^{-t} + \beta_1 e^{\sqrt{8}t} + \beta_2 e^{-\sqrt{8}t}, \quad \alpha_i, \beta_i \in \mathbb{R}. \]

Problem 11. \( x^{(6)} - 9x^{(3)} + 8x = 0 \)

Solution. The characteristic equations factors as \( (\lambda^3 - 1)(\lambda^3 - 8) \). The cube roots of 1 and 8 were both computed in Problem 4. The general solution is given by

\[ x(t) = \alpha_1 e^t + \alpha_2 e^{-\frac{1}{2}t} \cos \left( \frac{\sqrt{3}t}{2} \right) + \alpha_3 e^{-\frac{1}{2}t} \sin \left( \frac{\sqrt{3}t}{2} \right) + \beta_1 e^{2t} + \beta_2 e^{-t} \cos(\sqrt{3}t) + \beta_3 e^{-t} \sin(\sqrt{3}t), \quad \alpha_i, \beta_i \in \mathbb{R}. \]

Problem 13. \( x^{(6)} + 64x = 0 \)

Solution. The solutions to the characteristic equation are given in Problem 7. The general solution is given by

\[ x(t) = \alpha_1 \cos 2t + \alpha_2 \sin 2t + \beta_1 e^{\sqrt{3}t} \cos t + \beta_2 e^{-\sqrt{3}t} \sin t + \gamma_1 e^{-\sqrt{3}t} \cos t + \gamma_2 e^{-\sqrt{3}t} \sin t, \]

where \( \alpha_i, \beta_i, \gamma_i \in \mathbb{R}. \)