Math 4200, Problem set 1

Solutions

September 25, 2013

Problem 1. \( \dot{x} = ax + 3. \)

Solution.

Problem 2. \( \dot{x} = x^2 - ax. \)

Solution.
Problem 3. \( \dot{x} = x^3 - x + a. \)

Solution.

Problem 4. \( \dot{x} = x^2 - 1. \)

Solution. The integral curves are as follows:

This equation has two equilibria at \( x = \pm 1. \) To find the other solutions we use the method of separation of variables:

\[
\int \frac{dx}{x^2 - 1} = \int dt, \]

hence \(-\frac{1}{2} \log |x + 1| + \frac{1}{2} \log |x - 1| + C = t. \) This can be further simplified to

\[
\left| \frac{x - 1}{x + 1} \right| = ke^{2t}, \quad k = e^{-2C}. \tag{1}
\]

We now split into two cases depending on the sign of \( \frac{x - 1}{x + 1}. \)
If $x > 1$ or $x < -1$ then $\frac{x-1}{x+1} > 0$ and we can solve equation (1) for $x$ to obtain

$$x(t) = \frac{1 + ke^{2t}}{1 - ke^{2t}}.$$ 

If $x \in (-1, 1)$ then $\frac{x-1}{x+1} < 0$. Solving equation (1) for $x$ yields

$$x(t) = \frac{1 - ke^{2t}}{1 + ke^{2t}}.$$ 

Note that since $k = e^{-2C} > 0$ the solution $x(t)$ is bounded whenever $x_0 \in (-1, 1)$ and goes off to $\pm \infty$ in finite time whenever $|x| > 1$. 

\[ \Box \]

**Problem 5.** $\dot{x} = x^2 - 3x$.

**Solution.** The integral curves are as follows:

This equation has equilibria at $x = 0$ and at $x = 3$. We can solve this equation in exactly the same manner as we did in the previous problem. The solutions are the following.

$$x(t) = \frac{3}{1 - ke^{2t}}, \quad k > 0, \quad \text{for} \quad x_0 > 3 \text{ or } x_0 < 0,$$

and

$$x(t) = \frac{3}{1 + ke^{2t}}, \quad k > 0, \quad \text{for} \quad x_0 \in (0, 3).$$ 

\[ \Box \]
Problem 5. \( \dot{x} = \cos x \).

Solution. The integral curves are as follows:

This equation has equilibria at every point of the form \( x = \frac{3\pi}{2} + k\pi, \ k \in \mathbb{Z} \). Separation of variables gives \( |\sec x + \tan x| = ke^t, \ k > 0 \). Note that \( \sec x + \tan x \) is positive whenever \( \cos x \) is positive. In such case let us square both sides of equation

\[
\frac{\sin x + 1}{\cos x} = ke^t
\]

to obtain

\[
\frac{\sin^2 x + 2\sin x + 1}{\cos^2 x} = k^2 e^{2t}.
\]

Note that

\[
\frac{\sin^2 x + 2\sin x + 1}{\cos^2 x} = \frac{2\sin x + 2}{\cos^2 x} - 1 = \frac{2ke^t}{\cos x} - 1.
\]

Equating both right hand sides of the last two equations and solving for \( \cos x \) gives

\[
\frac{1}{\cos x} = \frac{k^2 e^{2t} + 1}{2ke^t},
\]

which can be solved for \( x \) yielding the solution

\[
x(t) = \sec^{-1} \left( \frac{k^2 e^{2t} + 1}{2ke^t} \right).
\]

The graph of this formula matches the diagram previously plotted.

A similar solution is obtained when \( \cos x \) is negative. \( \square \)