

**problem 3**

In order to show that addition is well defined, we must show that if

$$(1) \quad \frac{p}{q} \sim \frac{p'}{q'}, \quad \frac{r}{s} \sim \frac{r'}{s'} \quad \text{then} \quad \frac{p}{q} + \frac{r}{s} \sim \frac{p'}{q'} + \frac{r'}{s'}$$

By definition,

$$\frac{p}{q} + \frac{r}{s} = \frac{ps+qr}{qs}, \quad \frac{p'}{q'} + \frac{r'}{s'} = \frac{p's'+q'r'}{q's'}$$

Also by definition,

$$\frac{p}{q} \sim \frac{p'}{q'} \Leftrightarrow pq' = qp' \quad \frac{r}{s} \sim \frac{r'}{s'} \Leftrightarrow rs' = sr'$$

Hence, since we know (1), we get,

$$qs(p's' + q'r') = qp'ss' + qq'sr' = q'ps's + q'qs'r = q's'(ps + qr)$$

So, again by definition, we get

$$\frac{ps+qr}{qs} \sim \frac{p's'+q'r'}{q's'}$$

Which is what we wanted, so we are done. Multiplication is proven similarly.

**problem 5** We want to show that  $\{x_n\}, \{y_n\}$  equivalent Cauchy sequences  $\Leftrightarrow \{z_n\}$  is a Cauchy sequence, where  $z_{2n-1} = x_n$  and  $z_{2n} = y_n$ . Here I will only show  $\Rightarrow$

( $\Rightarrow$ ) Given  $n > 0$ , choose

$$m_1 \text{ s.t. } \forall k, l > m_1, |x_k - x_l| < 1/n$$

$$m_2 \text{ s.t. } \forall k, l > m_2, |y_k - y_l| < 1/2n$$

$$m_3 \text{ s.t. } \forall k > m_3, |x_k - y_k| < 1/2n$$

The first two choices are possible because  $\{x_k\}, \{y_k\}$  are Cauchy sequences, and the third choice is possible because they are equivalent. Now let  $m = 2 \cdot \max\{m_1, m_2, m_3\}$ . For  $k, l > m$  we have

$$\text{for } k, l \text{ odd } |z_k - z_l| = |x_{(k+1)/2} - x_{(l+1)/2}| < 1/n \text{ by (1)}$$

$$\text{for } k, l \text{ even } |z_k - z_l| = |y_{k/2} - y_{l/2}| < 1/2n < 1/n \text{ by (2)}$$

$$\text{for } k \text{ even, } l \text{ odd } |z_k - z_l| = |y_{k/2} - x_{(l+1)/2}| \leq |y_{k/2} - y_{(l+1)/2}| + |y_{(l+1)/2} - x_{(l+1)/2}| < 1/2n + 1/2n = 1/n \text{ by (2) and (3)}$$

Similarly, for  $k$  odd,  $l$  even, we have  $|z_k - z_l| < 1/n$

Since  $n$  was chosen arbitrarily, this proves that  $\{z_k\}$  is a Cauchy sequence.

**problem 10**

The following proof is somewhat more general than what is required:

Let  $A$  be an uncountable set, and let  $C$  be a countable subset of  $A$ . From last week, we know that  $A \setminus C$  is still uncountable, so in particular, we may choose a countable (infinite) subset of  $A \setminus C$ , which we will call  $B$ . We now construct a bijection  $f : A \setminus C \rightarrow A$  as follows: On  $A \setminus \{C \cup B\}$   $f$  is defined to be the identity, and on  $B$   $f$  is defined to be a bijection  $f : B \rightarrow B \cup C$ . We know that such a bijection exists since both  $B$  and  $B \cup C$  are countable.  $f$  thus defined is clearly a bijection, so  $|A \setminus C| = |A|$ .

**comments** For the most part it seemed that most people understood what was going on in this problem set. However, I would like to make two comments

on organizing proofs.

(1) When proving inequalities, it is generally best to string things together:

$$\begin{aligned} a < b & \qquad c \leq f \\ \downarrow & \\ e < f & \\ c > e & \\ \Rightarrow a < b & \end{aligned}$$

is unclear and prone to mistakes.

$$a = e < c \leq f = b$$

is much better.

(2) Proofs involving showing that things are small ( $< \epsilon, < 1/n$  for any  $\epsilon, 1/n$ ) should generally be organized as follows:

Given  $n > 0$  ( $\epsilon > 0$ ), choose  $m$  s.t. .... (give criteria for choosing  $m$ ) .... Now, for any  $k > m$  .....(give a proof that your choice of  $m$  works)

Again, proofs following this outline are easy to understand, and are less likely to be fatally flawed.