Hints for assignment 6

problem 3) Fix a point $x \neq x_0$, and let g_M be the "M" zoom of f at x_0 then,

$$\lim_{M\to\infty} g_M(x) = \lim_{M\to\infty} \frac{f(x_0 + x/M) - f(x_0)}{M} = \lim_{M\to\infty} x\frac{f(x_0 + x/M) - f(x_0)}{x/M} = xf'(x_0)$$

by the definition of the derivative, and the fact that $x/M \to 0$ as $M \to \infty$.

problem 6) We know that if $f(x) = x^k$, then $f'(x) = kx^{k-1} > 0$ for $x \in (0, \infty)$. Also, the image of f on $(0, \infty)$ is $(0, \infty)$, so by the Inverse Function Theorem, we can define f^{-1} on $(0, \infty)$. Also by the Inverse Function Theorem,

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{k(f^{-1}(x))^{k-1}} = \frac{(f^{-1}(x))^{1-k}}{k}$$

If we denote $f^{-1}(x)$ by $x^{1/k}$, this last expression becomes $(1/k)(x^{1/k})^{1-k}$, which is the usual formula.

Note that by what we have done so far, it is not clear if the usual algebra of exponents, such as $x^a x^b = x^{a+b}$ and $(x^a)^b = x^{ab}$ holds for fractional exponents, or is even meaningful in this context. Showing these properties is neither difficult, nor very exiting, so I will not do so here, and instead leave the final expression unsimplified.

problem 8) We first prove that if f' is a polynomial, then f is also a polynomial. For this, note that if $f'(x) = a_n x^n + ... + a_0$, then we can find a polynomial, $g(x) = (a_n/(n+1))x^{n+1} + ... + a_0 x$, such that $g' = f'$. Thus, $(g - f)' = g' - f' = 0$, and so the zero derivative theorem tells us that $g(x) - f(x) = c$, so $f(x)$ must be a polynomial. Now we show by induction on n that if $f^{(n)} = 0$, then f is a polynomial. For the base case, we just have $f = 0 \Rightarrow f$ is a polynomial, namely the polynomial 0. Now assume that $f^{(n-1)} = 0 \Rightarrow f$ is a polynomial. If $f^{(n)} = 0$, then $(f')^{(n-1)} = 0$, so by our induction hypothesis, f' is a polynomial, and so by the lemma above, f is a polynomial, and we are done.

problem 9) We define a function on $\mathbf{R}$ as $f(x) = d_-^2(x, A) \cdot d_+^2(x, A)$, where $d_-(x, A) = inf\{|x - a|; a \in A,\ a \leq x\}$, and $d_+(x, A)$ is defined similarly. If A is bounded, on the unbounded intervals of $C(A)$, we drop the undefined term in the definition of $f(x)$. Now, $f(x)$ is differentiable for all x. Pf:
If $x \in C(A)$, then x is in the interior of an interval, (a, b) where $f(x) = (x - a)^2(x - b)^2$, so f is differentiable at x. On the other hand, if $x \in A$,

then we compute:

$$\lim_{h\to 0_+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0_+} \frac{f(x+h)}{h} \leq \lim_{h\to 0_+} \frac{(x+h-x)^2(d_+^2(A,x+h))}{h} = \lim_{h\to 0_+} h{\cdot}d_+^2(A,x{+}h) = 0$$

where the last step follows since $d_+(A,x+h)$ is bounded as $h \to 0$ Similarly,

$$\lim_{h\to 0_-} \frac{f(x+h)-f(x)}{h} = 0$$

so f is differentiable for all x. Showing that the derivative is continuous involves computations similar to the above, and we (that is I) choose to omit them.