Lecture Notes for talk on Isoperimetric Inequality

Matthew Fontana
MATH 401
Talk to be given on 2/27/07 about 60 minutes

The Isoperimetric Problem

We will study only the planar case.

1. There are two statements of the Isoperimetric Problem:
   2. Among all figures in the plane (say, simple closed curves) with a given perimeter \( L \), which one has the greatest area \( A \)?

2. Among all figures in the plane with a given area \( A \), which one has the smallest perimeter \( L \)?

Fact: The circle is the figure that answers both of these questions.

Fact: Statements 1 and 2 of the Isoperimetric Problem are equivalent.

Proof: (By contradiction)

1. \( \Rightarrow \) 2: Fix a perimeter \( L \). Suppose we have a figure \( F \) of perimeter \( L \) that satisfies (1) but not (2). Say the area of the figure \( F \) that satisfies (1) is \( A \). Since \( F \) does not satisfy (2), there is a figure \( G \) with area \( A \) and perimeter \( L' \leq L \). Expand \( G \) so it has perimeter \( L \). Then \( G \) will have a larger area than \( A \). Hence \( F \) does not satisfy (1), a contradiction.
\( \Rightarrow \): Suppose we have a figure \( G \) with area \( A \), that satisfies \( \bullet \) but not \( \circ \). Let \( L \) be the perimeter of \( G \). Since \( G \) does not satisfy \( \circ \) there is a figure \( F \) with perimeter \( L \) and area \( A^k > A \). Shrink \( F \) so it has Area \( A \). Then \( F \) will have a perimeter \( L^k < L \). So \( G \) does not satisfy \( \circ \), a contradiction.

I will prove later that the circle satisfies both \( \bullet \) and \( \circ \) but for now I will prove the following fact:

**Theorem:** Suppose a figure \( F \) solves the isoperimetric problem. Then \( F \) is convex.

**Proof by Contradiction:** Suppose \( F \) is not convex. Then the convex hull of \( F \) will have a larger area and smaller perimeter than \( F \).

A way to construct the convex hull of a set \( S \) is by taking all of the convex combinations of points from \( S \): 

\[ \frac{1}{2} \xi_i + \frac{1}{2} \eta_j, \quad \xi_i, \eta_j \in S, \quad \forall i, j \in \mathbb{N}, \quad \forall i, j \leq 213. \]
Jacob Steiner gave some proofs that the circle solves the isoperimetric problem. I will present one of his proofs here.

**Theorem:** Any figure with maximal area must be a circle.

**Proof:** (Steiner’s four-hinge proof)

Take a figure with maximal area. Cut its perimeter in half with a line. Then the line will split the area in half as well.

If it did not, then reflect the half with the larger area about the line, and the figure with the larger half and its reflection will have the same perimeter as before, but a larger area. This would contradict the fact that the original figure was one with maximal area.

Now suppose that one of the halves is not a semicircle. (*) Then we can find a point on the boundary of the half where the lines drawn from the points on the intersection of the boundary of the figure and the line of symmetry to the chosen point on the boundary do not form a right angle.
We can increase the area of the figure in the following way: "Glue" the two parts of the figure that are not inside the triangle to the sides of the triangle that each part is "attached" to. Then slide one or both endpoints along the line of symmetry until the lines form a right angle. Then we increase the area of the triangle increased, and the area of the other parts of the figure remained the same. So the area of the whole figure increased while its perimeter remained the same, a contradiction to the fact that the figure was one of maximal area. So each of the halves is a semicircle and the figure is a circle.

I feel that statements 1 and 2 should be proved.

1: Suppose we have a semicircle of radius r. Then draw a line from the intersection of the boundary of the circle part and the diameter to the boundary point P. Then the two lines will meet at a right angle.
Proof:

Given a semicircle with diameter $AB$, center $O$, boundary point $P$, and line segments $AP$ and $BP$.

Draw a line segment from $P$ to $O$: $OP$. Since $OP$, $AO$, and $OB$ are all radii of circle $O$, the circle they all have length $r$, and are therefore congruent.

Since $\triangle APO$ is an isosceles triangle, $\angle OAP$ and $\angle APO$ are congruent.

Since $\triangle BPO$ is an isosceles triangle, $\angle OBP$ and $\angle BPO$ are congruent.

Let $m\angle OAP = a$ and let $m\angle OBP = b$.

Let $m\angle AOP = w$.

Since the 3 angles in a triangle add up to $180^\circ$ and since all of the angles on a line add up to $180^\circ$:

1. $a + a + m\angle AOP = 180$
2. $b + b + m\angle BOP = 180$
3. $m\angle AOP + m\angle BOP = 180$
equations \((1 + 2) - (3)\) yields
\[ 2a + 2b = 180 \]
\[ \Rightarrow a + b = 90 \]

Since \(a\), \(\therefore m \angle APB = m \angle APQ + m \angle PQB = a + b = 90\) and lines \(AP\) and \(BP\) meet at a right angle.

Given a triangle with fixed side lengths \(a\) and \(c\), the triangle with the largest area is a right triangle.

\[ \sin A = \frac{x}{c} \Rightarrow x = c \sin A \]

The area of the triangle \(B = \frac{1}{2} \cdot \text{base} \cdot \text{height}\)
\[ \Rightarrow \text{Area} = \frac{1}{2} \cdot b \cdot (c \sin A) = \frac{1}{2} bc \sin A. \]

Since the sides are constant, we can change the area of the triangle by changing \(A\) from anywhere from 0° to 180°.

Since \(\sin A\) has a maximum at \(A = 90°\) \([\text{when } A \in (0, 180°)]\) the area of the triangle is maximized at \(A = 90°\)
\[ \Rightarrow \text{Maximum area} = \frac{1}{2} bc \sin 90° = \frac{1}{2} bc. \]
Remark: There is one flaw in Steiner's proof: it assumes that given a perimeter $L$, there is a figure that has a maximum area $A$. Given such a figure exists, Steiner's proof shows that the figure is a circle.

There are ways to fix Steiner's proof by proving the existence of a figure with maximal area.

Some proofs of existence can be found in Viktor Blasjö's paper "The Isoperimetric Problem".

One main idea is compactness.

\text{(in } \mathbb{R}^2 \text{)}

Definition: A set $S$ is compact if and only if for every sequence of points in $S$, there is a subsequence of points that converges to a point in $S$.

A fact from analysis is that a set $S \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

A compactness argument for existence is the following (see p. 344 of Blasjö's paper):

Fix a perimeter $L$. Consider a sequence of closed curves of perimeter $L$ that are bounded inside a square in the plane. Since any sequence of curves of perimeter $L$ cannot converge to a curve with a different perimeter, the set of closed curves in the square of length $L$ in the square is closed and bounded, and therefore compact. Hence there is a subsequence of curves that converges to a "limit" curve, (or perhaps by taking a subsequence of a subsequence of a subsequence etc.).

There are many details missing from my argument above, but the "limit" curve is what we are looking for.
Steiner Symmetrization

Steiner gives a proof that the circle solves the isoperimetric problem using a method called Steiner symmetrization. I will not present the proof here, but a description of Steiner symmetrization in page 64 of Glen Bracken's paper is:

"3. Given a bounded convex region \( R \) and a line \( l \), the axis of symmetrization, we define \( R_1 \) the symmetrized image of \( R \) with respect to \( l \) as follows. Consider lines \( r \) which are perpendicular to \( l \) and cut \( R \). Let \( r' \) be the segment of \( r \) which has the intersection of \( r \) with \( R \). Let \( r_1 \) be the segment of \( r \) which has the same length as \( r' \) and has its midpoint on \( l \). We then define \( R_1 \) as the union of all these line segments \( r_1 \) (see Fig. 2). Note: \( R_1 \) is convex.

\[\text{Figure 2}\]
I have presented the following theorem:

Isoperimetric Theorem: Among all figures in the plane (simple closed curves) with a given perimeter \( L \), the circle has the greatest area.

This motivates:

Isoperimetric Inequality: Suppose we have a figure in the plane (simple closed curve) with perimeter \( L \) and area \( A \). Then \[ A \leq \frac{L^2}{4\pi}, \] with equality if and only if the figure is a circle.

In fact, the Isoperimetric theorem and Isoperimetric inequality are equivalent.

Proof: Let \( F \) be a figure in the plane with area \( A \) and perimeter \( L \).

Isoperimetric Theorem \( \Rightarrow \) Isoperimetric Inequality:
If \( F \) is a circle, denoting with radius \( r \) then
\[ L = 2\pi r \quad \text{and} \quad A = \pi r^2 \]
\[ \Rightarrow \frac{L^2}{4\pi} = \frac{(2\pi r)^2}{4\pi} = \frac{4\pi^2 r^2}{4\pi} = \pi r^2 = A \]

If \( F \) is not a circle, then there is a circle \( G \) with perimeter \( L \) and area \( A^* \geq A \) by the Isoperimetric Theorem. Then
\[ A \leq A^* = \frac{L^2}{4\pi} \quad \text{since} \ G \ \text{is a circle}. \]
Isoperimetric Inequality => Isoperimetric Theorem:

If \( F \) is a circle, then \( A = \frac{L^2}{\pi} \) which is (the largest \( \frac{\pi}{4} \)).

Otherwise, possible area for \( F \) for a fixed \( L \) by the Isoperimetric Inequality.

If \( F \) is not a circle, then \( A \leq \frac{L^2}{\pi} \). Let \( G \) be a circle with perimeter \( L \). Then the area of \( G \) is \( A^* = \frac{L^2}{4\pi} \). Therefore, \( A < A^* = \frac{L^2}{4\pi} \).

So the figure of a fixed perimeter \( L \) that has the largest area is the circle. \( \Box \)
Here is a proof of the Isoperimetric inequality for simple closed curves in the plane (see pgs. 550-551 in Blasja's paper)

**Schmidt's Projection Proof:** Let \( \gamma \) be a simple closed curve in the plane. Parametrize \( \gamma \) by arclength, and use a unit speed parameterization. Then \( \gamma(t) = (x(t), y(t)) \) with \( \frac{d}{dt} \gamma(t) = \sqrt{(x'(t))^2 + (y'(t))^2} = 1 \). Let \( L \) be the arclength of \( \gamma \).

Now construct a circle whose width in the \( x \)-direction is the same as the width of \( \gamma(t) \) in the \( x \)-direction.

Now project the \( y \) coordinate of \( \gamma(t) \), keeping the \( x \) coordinate fixed, onto the circle, so \( (x(t), y(t)) \rightarrow (x(t), f(t)) \) where \( f(t) \) is the \( y \) coordinate of the circle.
Now consider the vector \((x, y)\) and \(\left(\frac{\partial y}{\partial t}, -\frac{\partial x}{\partial t}\right)\).

To save space, let \(\frac{\partial t}{\partial t} = \frac{1}{r}, \frac{\partial x}{\partial t} = \frac{1}{r}\).

By the Cauchy-Schwarz inequality,

\[
\langle (x, y), (-y, x) \rangle \leq \sqrt{x^2 + y^2} \cdot \sqrt{y^2 + x^2} = \sqrt{x^2 + y^2} \cdot \sqrt{y^2 + x^2}
\]

\[
\Rightarrow x^2 - y^2 \leq \sqrt{x^2 + y^2} \cdot \sqrt{y^2 + x^2} = \sqrt{(x^2 + y^2) + (y^2 + x^2)}
\]

Let \(\sqrt{x^2 + y^2} = r\), the radius of the circle.

\[
\Rightarrow x^2 - y^2 \leq r
\]

since we used a unit speed parameterization of \(y(t)\).

We get equality if and only if \(y\) is a circle. Why?

Let \(C\) be the parameterization of the circle of radius \(r\)

\[
\gamma(t) = (r \cos t, r \sin t) = (x, y)
\]

\[
\Rightarrow \gamma'(t) = (-r \sin t, r \cos t) = (y, x)
\]

\[
\Rightarrow \gamma''(t) = (-r \cos t, -r \sin t) = (-x, -y) = -(x, y)
\]

\[
\gamma'(t) = \frac{\partial y}{\partial t} \gamma(t) - \frac{\partial x}{\partial t} \gamma(t) = (x, y) \text{ normal to the circle, although the normal vector is usually pointing inside the curve.}
\]

Since \(\gamma'(t) = \frac{\partial y}{\partial t} \gamma(t) - \frac{\partial x}{\partial t} \gamma(t)\) is the tangent vector to the curve \(\gamma(t)\), we rotate the tangent vector \(90^\circ\) clockwise (or \(270^\circ\) counterclockwise) to get a vector that is normal to the curve. The vector after the rotation is \((-y, x)\).
Now, $(y, -x) = (x, y)$ for all $t$, if and only if $\gamma(t)$ is a circle.

If $\gamma(t)$ is not a circle, then there will be a $t_0$ such that $(x(t_0), y(t_0); y(t_0), -x(t_0)) < r$

(In fact, since $\gamma(t)$ and the circle are both continuous, there will be an interval $(t_1, t_2) \subseteq [0, L]$ where $t_0 \in (t_1, t_2)$ and $(x(t_1), y(t_1); y(t_1), -x(t_1)) < r \quad \forall \quad t_0 \in (t_1, t_2)$)

This inequality tells us

$$x y \leq y x + r$$

$$= \int_{t_0}^{t_1} x y \, dt \leq \int_{t_0}^{t_1} x \, dt + \int_{t_0}^{t_1} y \, dt$$

Now we apply Green's Theorem.

Green's Theorem: Let $C$ be a simple closed curve and $D$ its interior and boundary. Let $P: \mathbb{D} \to \mathbb{R}, \; Q: \mathbb{D} \to \mathbb{R}$ be $C^1$ functions. Then

$$\oint_C P \, dx + Q \, dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy$$

where $C^+$ is the positive orientation of $C$. 

$\oint_{C^+}$
From Green's Theorem we can derive the following:
\[
\frac{1}{2} \oint_C x \, dy - y \, dx = \frac{1}{2} \iint_D \left( \frac{\partial}{\partial y} (x) - \frac{\partial}{\partial x} (y) \right) \, dx \, dy
\]
\[
= \frac{1}{2} \iint_D \left( 1 + 1 \right) \, dx \, dy = \iint_D \, dx \, dy = \text{Area of } D
\]

Also,
\[
\int_0^L x \, dt = \int_0^L x \, dy = \iint_D \left( \frac{\partial}{\partial y} (x) - \frac{\partial}{\partial x} (y) \right) \, dx \, dy = \iint_D \, dx \, dy = \text{Area of } D
\]
\[
\int_0^L y \, dt = \int_0^L y \, dy = \iint_D \left( \frac{\partial}{\partial y} (y) - \frac{\partial}{\partial x} (x) \right) \, dx \, dy = \iint_D \, dx \, dy = \left( \text{Area of } D \right)
\]

For a curve that is parameterized by for
\[0 \leq t \leq L,
\]

Therefore,
\[
\int_0^L x \, dy = \int_0^L x \, dt + \int_0^L y \, dt
\]
\[
\Rightarrow \text{Area } \delta \subseteq \bigcup_{i=1}^n \text{Area of the circle}
\]

The circle has area \(\pi r^2\), and let \(A = \text{area of } \delta\)

Then \(A \leq Lr - \pi r^2\)

Note that \(Lr - \pi r^2 = Lr - \pi r^2 + \frac{L^2 - L^2}{\frac{4\pi}{4\pi}} = \frac{1}{\frac{4\pi}{4\pi}} \left( L^2 - \pi r^2 \right)
\]
\[
\Rightarrow Lr - \pi r^2 = \frac{L^2 - \pi r^2}{\frac{4\pi}{4\pi}} (L - 2\pi r)^2
\]

Since \((L - 2\pi r)^2 \geq 0\), with equality if and only if
\[L = 2\pi r,
\]
\[A \leq Lr - \pi r^2 \leq \frac{L^2}{\frac{4\pi}{4\pi}} \text{ with equality if and only if } \delta \text{ is a circle.} \]
References

A lot of the material for this talk was taken from

"Blasjö, Viktor. "The Isoperimetric Problem." The American
Vol. 112, Iss. 6, p. 526 (41 pages).

Other resources that I found helpful for understanding
the material on the Isoperimetric problem are:

Bredon, Glen E. "The Isoperimetric Problem in the Plane."
Mathematics Magazine, Vol. 30, No. 2 (Nov. - Dec. 1956);
pp. 63-69.

It can be accessed on the JSTOR archive:
<http://www.jstor.org/stable/2690611>

"Isoperimetric Theorem and Inequality," February 24, 2007
<http://www.cut-the-knot.org/do_you_know/isoperimetric.shtml>