

segment is a positive segment for one of these polygons and a negative segment for the other. The sides of  $Q_1, Q_2, \dots, Q_l$  parallel to  $a$  and lying inside  $Q$  are divided into positive and negative segments such that each positive segment corresponds to a negative segment of equal length (in fact, coincides with it), and conversely. The algebraic sum  $c_3$  of all such segments is zero. From this and (4), it follows that

$$d - d_1 = 0,$$

that is,  $d = d_1$ . Therefore, each side of the convex polygon is equal to the side parallel to it. Then, by Theorem 2, section 13,  $Q$  has a center of symmetry.

The proof of the theorem for polyhedra proceeds entirely analogously, where we consider faces and parts of faces instead of sides and segments of sides, and areas instead of lengths.

In conclusion we give without proof two theorems. Theorem 4 is due to A. D. Aleksandrov.

**THEOREM 4.** *If every face of a convex polyhedron  $Q$  has a center of symmetry, then the polyhedron  $Q$  itself also has a center of symmetry.*

**THEOREM 5.** *If every face of a convex polyhedron  $Q$  has a center of symmetry, then  $Q$  can be divided into parallelepipeds.*

## 15. THE GREATEST CENTRAL-SYMMETRIC CONVEX FIGURE IN A LATTICE OF INTEGERS; MINKOWSKI'S THEOREM

We shall consider systems of straight lines in the plane that divide the plane into strips of equal width. Two such systems of parallel lines divide the plane into congruent parallelograms and form a plane network or lattice. The vertices of these parallelograms are called *lattice points*, and the parallelograms themselves are called *the fundamental parallelograms* of the lattice. Lattice points are the points of intersection of the straight lines of the two systems (Fig. 62).

Analogously, in three-dimensional space we consider systems of parallel planes dividing the space into layers of equal thickness. Three such systems divide the

space into congruent parallelepipeds and form a three-dimensional network or lattice. The vertices of these parallelepipeds (in which intersect planes of all three systems) are *lattice points* and the parallelepipeds themselves are the *fundamental parallelepipeds* of the lattice.

**DEFINITION.** *A lattice as described above in the plane or in three-dimensional space is called a lattice of integers.*

We shall assume that the fundamental parallelograms (parallelepipeds) of the lattice have area (volume) equal to 1.

For simplicity, we shall consider a plane lattice of integers for which the fundamental parallelograms are squares (Fig. 63). In this

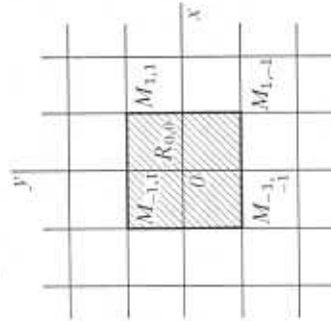


Fig. 63

case the straight lines of the two systems are perpendicular. Choose a lattice point  $O$  for the origin. There is a line of each of the two systems passing through this point. Take one of these lines to be the axis of abscissas and the other for the axis of ordinates, and take the length of a side of a fundamental square for the unit of length. Then all lattice points  $(k, l)$  have integral (positive, negative, or zero) coordinates  $k$  and  $l$ . We shall denote a lattice point  $(k, l)$  by  $M_{k,l}$ . If we subject the lattice to an integral translation, that is, translate by a vector  $\vec{OA}$  where  $O = M_{0,0}$  (the origin) and  $A = M_{p,r}$ , then each lattice point  $M_{k,l}$  goes into point  $M_{k+p,l+r}$  and the entire lattice is carried onto itself.

Analogous definitions are made for a three-dimensional lattice. *Extremal problems* are those that concern lines and other figures for which some quantity assumes the least value. They play an important role in mathematics. A simple example of such a problem is to find the shortest curve connecting two points of a plane. Such

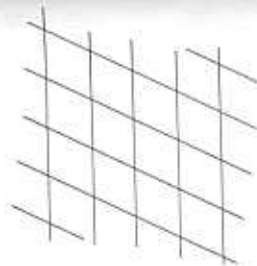


Fig. 62

a curve, obviously, is a straight line segment. The reader may look through our book *Shortest Paths*<sup>1</sup> for examples of such problems. There is a large selection of interesting extremal problems about convex figures in the book by I. M. Yaglom and V. G. Boltyanskii mentioned at the end of Chapter 1.

We have previously considered the simplest of such problems, that is, finding the smallest convex figure containing three non-collinear points  $A, B, C$ . It turned out to be the triangle  $ABC$ . In sections 32 and 40 we shall consider the so-called *isoperimetric problem*, that is, the problem of finding the plane figure having the greatest area for a boundary of given length.

We shall now give an interesting extremal problem connected with the arrangement of central-symmetric convex figures in a lattice of integers. A plane convex figure (convex solid)  $Q$  is said to *cover* a point  $P$  if  $P$  is an interior point of  $Q$ . Suppose that we are given a plane lattice of integers and a convex central-symmetric figure  $Q$  with its center at one of the lattice points and not covering any other lattice point. G. Minkowski stated and solved the problem of finding the maximal possible area of such a figure and the analogous problem for a three-dimensional lattice. The following theorem gives the solution.

**MINKOWSKI'S THEOREM.** (a) *The greatest area of a plane central-symmetric convex figure  $Q$  whose center of symmetry coincides with one of the lattice points of an integral lattice and which does not cover any other lattice points is equal to 4.*

(b) *The greatest volume of a three-dimensional convex central-symmetric solid whose center of symmetry coincides with one of the lattice points of an integral lattice and which does not cover any other lattice point is equal to  $8/3$ .*

*Proof of (a).* Let  $R_{0,0}$  denote the square with vertices at the lattice points  $M_{1,1}, M_{-1,1}, M_{-1,-1}, M_{1,-1}$  and center at the point  $O = M_{0,0}$  (Fig. 63). The area of this square is equal to 4, and the square  $R_{0,0}$  does not cover any lattice points other than its center  $O$ . If we translate  $R_{0,0}$  by vector  $\overrightarrow{OM_{m,n}}$ , then it goes over to square  $R_{m,n}$  with area also equal to 4 and with center at lattice point  $M_{m,n}$ .

<sup>1</sup> L. A. Lyusternik, *Shortest Paths: Variational Problems* (New York-London: Pergamon Press, 1964).

<sup>2</sup> The theorem has been proved for spaces with  $n$  dimensions. The maximal  $n$ -dimensional volume under the conditions of the theorem is  $2^n$  (for  $n = 2$  and  $n = 3$ ,  $2^n$  equals 4 and 8, respectively).

and vertices at lattice points  $M_{m+1,n+1}, M_{m-1,n+1}, M_{m-1,n-1}$ , and  $M_{m+1,n-1}$ .

Now let  $Q_{0,0}$  be a convex central-symmetric figure with center at  $O = M_{0,0}$ , covering no lattice points other than its center. Let  $Q_{m,n}$  denote the figure obtained by translating  $Q_{0,0}$  by vector  $\overrightarrow{OM_{m,n}}$ . We must prove that the area of  $Q_{0,0}$  (and, therefore, of each of the figures  $Q_{m,n}$ ) does not exceed 4 (the area of  $R_{0,0}$ ).

We consider the set of squares  $R_{2m,2n}$  with centers at points  $M_{2m,2n}$  with even integral coordinates. The entire plane is thus divided into an infinite set of such squares, which fill the plane and do not overlap each other. Two figures are said to *overlap* each other if they have common interior points. We also consider the set of all figures  $Q_{2m,2n}$ . We shall now prove the following:

**LEMMA.** *If  $Q_{0,0}$  does not cover any lattice points other than its center, then the figures  $Q_{2m,2n}$  do not overlap each other.*

*Proof.* Suppose that the figures  $Q_{2m,2n}$  and  $Q_{2m',2n'}$  with centers at the points  $M_{2m,2n} = M$  and  $M'_{2m',2n'} = M'$  overlap each other, that is, have a common interior point  $C$ . We consider the two possible cases:

Case 1. The common interior point  $C$  does not lie on the line  $MM'$  (Fig. 64).

Fig. 64

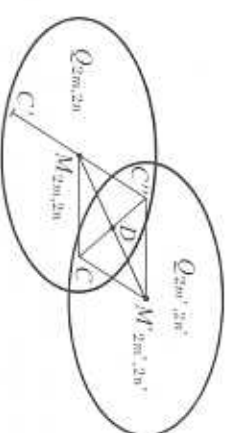


Figure  $Q_{2m,2n}$  is obtained from  $Q_{2m',2n'}$  by translation by the vector  $\overrightarrow{MM'}$ ; the segment  $M'C$  lies entirely inside  $Q_{2m',2n'}$  (since  $M'$  and  $C$  lie inside the convex figure  $Q_{2m',2n'}$ ). If we translate the figure  $Q_{2m',2n'}$  by vector  $\overrightarrow{MM'}$  so that it coincides with  $Q_{2m,2n}$ , then the segment  $M'C$  goes into an equal and parallel segment  $M'C'$  lying entirely inside  $Q_{2m,2n}$ . Let  $M'C''$  denote the segment equal and parallel to  $M'C'$  but lying on the opposite side of point  $M$ . This segment is also equal and parallel to  $M'C$ . Since  $Q_{2m,2n}$  has a center of symmetry at point  $M$ , segment  $M'C''$ , as well as segment  $M'C'$ , is symmetric to  $M'C$  with respect to point  $M$ , lies entirely in  $Q_{2m,2n}$ .

The quadrilateral  $MCM'C''$  has a pair of equal and parallel sides  $MC$  and  $MC''$ , so that it is a parallelogram. Its diagonals  $MM'$  and  $CC''$  intersect at a point  $D$ , their common midpoint. Since the points  $C$  and  $C''$  both lie inside the convex figure  $Q_{2m,2n}$ , the entire segment  $CC''$  lies inside this figure. In particular, the point  $D$  lies inside the figure  $Q_{2m,2n}$ . As the mid-point of the segment  $MM'$ , point  $D$  has coordinates

$$x = \frac{2m + 2m'}{2} = m + m',$$

$$y = \frac{2n + 2n'}{2} = n + n',$$

that is, point  $D$  has integral coordinates and consequently is a lattice point. Thus, the figure  $Q_{2m,2n}$  covers a lattice point different from its center  $M_{2m,2n}$ , contrary to the assumption of the lemma.

Case 2. The common interior point  $C$  lies on the segment  $MM'$  (Fig. 65).

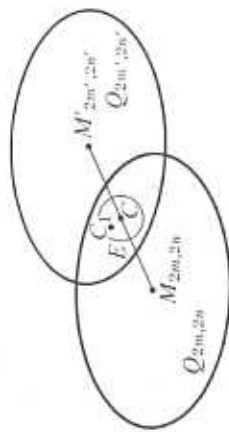


Fig. 65

Then there exists a circle  $E$  with center at  $C$  which lies entirely inside both figures. This circle contains a point  $C_1$  not lying on the segment  $MM'$  but which is an interior point of both figures  $Q_{2m,2n}$  and  $Q_{2m',2n'}$ . We have now reduced this case to the preceding case.

If figures  $Q_{2m,2n}$  and  $Q_{2m',2n'}$  have a common interior point, then  $Q_{2m,2n}$  contains at least one lattice point besides  $M_{2m,2n}$ , and this contradicts our assumption. Thus, the lemma is proved.

*Proof of (a) (continued).* If the figure  $Q_{0,0}$  lies entirely inside the square  $R_{0,0}$ , then its area does not exceed the area of  $R_{0,0}$ , that is, 4 in this case. But if the figure  $Q_{0,0}$  lies only partly in the square  $R_{0,0}$  (Fig. 66), then  $Q_{0,0}$  is divided into several parts:

Part  $A_1$  is the intersection of  $Q_{0,0}$  and  $R_{0,0}$ , and parts  $A_2, A_3, \dots, A_k$  are the intersections of  $Q_{0,0}$  and squares  $R_{2m,2n}$  different from

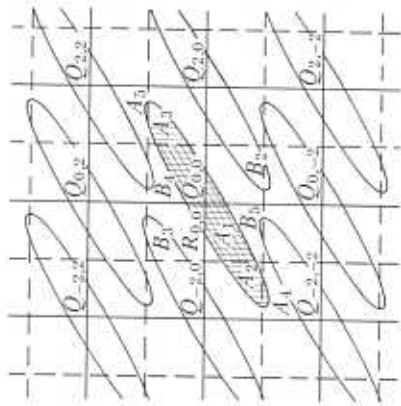


Fig. 66

$R_{0,0}$ . Consider one such part  $A_i$  of the figure  $Q_{0,0}$  ( $i = 2, 3, \dots, k$ ). It lies in some square  $R_{2m,2n}$  with center of symmetry  $M = M_{2m,2n}$ . The square  $R_{-2m,-2n}$  is symmetric to  $R_{2m,2n}$  with respect to the origin and has center of symmetry  $M' = M_{-2m,-2n}$ . Translate the plane by the vector  $\overline{MO} = \overline{OM'}$ . Under this translation the square  $R_{2m,2n}$  goes into the square  $R_{0,0}$ , and the square  $R_{0,0}$  into the square  $R_{-2m,-2n}$ . The figure  $Q_{2m,2n}$  with center at  $M_{2m,2n}$  goes into the figure  $Q_{0,0}$ , and  $Q_{0,0}$  into  $Q_{-2m,-2n}$  with center at  $M'$ . Figure  $A_i$ , the intersection of  $Q_{0,0}$  and  $R_{2m,2n}$ , goes into figure  $B_i$ , the intersection of  $Q_{-2m,-2n}$  and  $R_{0,0}$ .

Thus, in the square  $R_{0,0}$  we find a part  $B_i$  of the figure  $Q_{-2m,-2n}$  congruent to the part  $A_i$  of the figure  $Q_{0,0}$ . Similarly, parts  $A_2, A_3, \dots, A_k$  of figure  $Q_{0,0}$  are congruent to parts  $B_2, B_3, \dots, B_k$  of various figures  $Q_{2m',2n'}$  (different from  $Q_{0,0}$  and from each other). Since the various figures  $Q_{2m',2n'}$  do not overlap each other, then their parts  $B_i$  lying inside  $R_{0,0}$  also do not overlap each other and do not overlap  $A_1$ . The figures  $A_1, B_2, B_3, \dots, B_k$  cover either the entire square  $R_{0,0}$  or a part of it, and do not overlap each other. The sum of their areas does not exceed the area of  $R_{0,0}$ , that is, 4. But each part  $A_i$  of figure  $Q_{0,0}$  has the same area as the corresponding  $B_i$ . Then the sum of the areas of all the parts  $A_1, A_2, A_3, \dots, A_k$  of figure  $Q_{0,0}$  is equal to the sum of the areas of  $A_1, B_2, B_3, \dots, B_k$ , and consequently does not exceed 4. The area of  $Q_{0,0}$  is equal to the sum of the areas of its parts  $A_i$ . Hence, it is not greater than 4. Thus, Minkowski's theorem is proved for the plane case, (a).

In the preceding paragraph we saw that the area of the figure  $Q_{0,0}$  is equal to the sum of the areas of the parts  $A_1, B_2, B_3, \dots, B_k$  of the figures  $Q_{2m,2n}$  lying in the square  $R_{0,0}$ .

Because of this, we have the following:

- (1) If the area of  $Q_{0,0}$  is less than 4, then the parts  $A_1, B_2, B_3, \dots, B_k$  of the figures  $Q_{2m,2n}$  do not completely fill the square  $R_{0,0}$ .
- (2) If the area of  $Q_{0,0}$  is equal to 4 (that is, equal to the area of  $R_{0,0}$  itself), then these parts of the figures  $Q_{2m,2n}$  completely fill the square  $R_{0,0}$  (without gaps). Also, the parts of the figures  $Q_{2m,2n}$  will completely fill any square  $R_{2m,2n}$ . And since the entire plane is divided into these squares, the figures  $Q_{2m,2n}$  cover the entire plane without gaps and without overlapping each other, as was shown above.

We have obtained the following:

(a'). *If a convex figure  $Q_{0,0}$  has the maximum area, 4, then the figures  $Q_{2m,2n}$  fill the entire plane without gaps. Conversely, if the convex figures  $Q_{2m,2n}$  with centers at the lattice points having even integral coordinates cover the whole plane without gaps and without overlapping each other, then the area of each is 4.*

Thus, the problem of determining the figures  $Q_{0,0}$  having maximum area and the problem of determining the figures  $Q_{2m,2n}$  covering the plane without gaps and without overlapping are equivalent.

A corresponding development can be given in the three-dimensional case. Let  $R_{0,0,0}$  denote a cube with center at the origin, and edges 2 units long and parallel to the coordinates axes. Its vertices lie at the lattice points for which each of the coordinates is equal to 1 or  $-1$ . The interior of the cube  $R_{0,0,0}$  contains only one lattice point, that is, its center  $O$ .

Let us consider for a moment a convex solid  $Q_{0,0,0}$  with center at the origin and whose interior contains no lattice point other than its center. By reasoning entirely analogous with that above, we can prove the following statements in order:

1. The preceding lemma for the three-dimensional case: Let  $Q_{2l,2m,2n}$  denote one figure with center of symmetry at the lattice point with coordinates  $2l, 2m, 2n$  obtained from  $Q_{0,0,0}$  by translation. If  $Q_{0,0,0}$  does not cover any lattice points other than its center, then the solids  $Q_{2l,2m,2n}$  do not overlap each other.

2. Part (b) of Minkowski's theorem: Inside cube  $R_{0,0,0}$  are found part  $A_1$  of solid  $Q_{0,0,0}$  and parts  $B_2, B_3, \dots, B_k$  of other solids  $Q_{2l,2m,2n}$ . The sum of the volumes of these parts  $A_1, B_2, B_3, \dots, B_k$  is equal to the volume of solid  $Q_{0,0,0}$ . On the other hand, this volume does not exceed the volume of  $R_{0,0,0}$ , that is,  $2^3 = 8$ .

3. (b'), corresponding to (a') above: If solid  $Q_{0,0,0}$  has the maximal possible volume (equal to 8), then the solids  $Q_{2l,2m,2n}$  fill the entire three-dimensional space without gaps. Conversely, if the solids  $Q_{2l,2m,2n}$  fill three-dimensional space, then these solids have the greatest possible volume, 8.

## 16. FILLING THE PLANE AND SPACE WITH CONVEX FIGURES

We have seen in section 15 that the maximal central-symmetric figures  $Q_{2m,2n}$  in an integral lattice fill the plane without gaps. Now we shall determine the form of these figures.

Since the figures  $Q_{2m,2n}$  do not overlap each other, and, on the other hand, do not leave any gaps, they must adjoin each other, that is, have common parts of their boundaries. Since a common part of the boundaries of bounded convex figures can be only a straight line segment, the boundary of each of the figures  $Q_{2m,2n}$  consists of several straight line segments. Thus, all the figures  $Q_{2m,2n}$  are polygons. Since the polygons  $Q_{2m,2n}$  are central-symmetric, they have an even number of sides (the boundary of such a polygon may be divided into several pairs of symmetric sides). Obviously, the number of sides of  $Q_{2m,2n}$  is not less than 4.

Two cases may occur in covering the plane with the polygons  $Q_{2m,2n}$ :

Case 1. At least one side of the polygon  $Q_{0,0}$  adjoins two or more sides of other polygons  $Q_{2m,2n}$ .

Case 2. Each side of polygon  $Q_{0,0}$  adjoins one side of one of the neighboring polygons.

First we shall examine Case 1.

Let us establish a positive direction on the boundaries of polygon  $Q_{0,0}$  and all polygons  $Q_{2m,2n}$ , that is, the direction in which we would travel if we were to make a counterclockwise journey around the polygon. Then a common side of two polygons is assigned opposite directions by the two polygons.