This exam naturally divides into two sections. The first section is somewhat random, and hopefully it is fun and interesting. It is worth 70% of the exam grade. The second section of the exam is worth the remaining 30%, and focusses primarily on group actions. In the second section we work through a proof of a partial converse to Lagrange’s theorem. The resulting theorem we derive is the first of three beautiful sister-theorems. If you ask, I will discuss these with you when you hand in the exam.

Please write out your solutions cleanly. On any question in the second section of the exam, you may assume you have correctly completed the previous questions in that section.

You may not receive help from or work with another person while working on this exam. However, you are perfectly welcome to use any passive web page or text you wish.

Some notation: We will generally use the symbol $e$ to represent the identity element of any group under discussion, and if $X$ is a set, we will use $|X|$ to represent the cardinality or size of $X$.

Below is a diagram which has nothing directly to do with this course.
1 Random Thoughts

1. Consider the following set $\Gamma$ of permutations in $S_4$.

$$\Gamma = \{e, (1234), (13)(24), (14)(23), (12)(34), (13), (24)\}$$

This forms a subgroup of $S_4$, which is an isomorphic copy of $D_4$ in $S_4$ (consider $r \mapsto (1234)$ and $f \mapsto (14)(23)$ where $r$ and $f$ are our usual generators of $D_4$). We say that we have found a realization of $D_4$ as a group of permutations. Find a realization of the quaternion group as a group of permutations.

2. Consider the planar object diagrammed below, which we will call a four-tri.

![Four-tri Diagram](image)

This four-tri is cleverly constructed by God-like beings so that at any time, one can carry out any of the following motions (symmetries).

(a) Rotate the whole whole four-tri by 120° in either direction.
(b) Rotate the small triangle in the lower right corner by 120° in either direction.
(c) Rotate the center triangle by 120° in either direction.
(d) Flip the small triangle in the lower left corner over on its vertical axis.

Using these basic motions as generators, how many different symmetries does the four-tri have? Bonus: find a presentation for the four-tri symmetry group.

3. Let $B_3$ represent the braid group on three strands. Consider $P_3$, the subgroup of $B_3$ which consists of all braids which leave the beginning and ending of each strand in the same location.

(a) Prove that $P_3$ is a normal subgroup of $B_3$ (it is given that it is a subgroup).
(b) Let $Q = B_3/P_3$ be the quotient group of left cosets of $P_3$ in $B_3$. Prove that $Q$ is isomorphic to $S_3$. (Hint: show that $\sigma_1$ and $\sigma_1^{-1}$ are in the same left coset of $P_3$ in $B_3$.)

4. Suppose $G$ is a group with subgroups $H$ and $K$, where $H$, $K$, and $G$ satisfy the following properties.

(a) $H \cap K = \{e\}$.
(b) $H \triangleleft G$ and $K \triangleleft G$
(c) $G = HK$.

prove that $G \cong H \times K$. 


5. Let \( A = \{ g : \mathbb{R} \to \mathbb{R} \mid g(x) = ax + b, \text{where } a, b \in \mathbb{R} \text{ and } a > 0 \} \),
represent the positive slope affine group (under the operation of composition of functions).

(a) What is the center of \( A \)?
(b) Prove that the set \( T \) of lines with slope one is a normal subgroup of \( A \).

6. Consider the set \( \mathcal{F} \) of all functions from \( \mathbb{R} \) to \( \mathbb{R} \) which move only finitely many points.

(a) Prove that under the operation of composition of functions, \( \mathcal{F} \) is a group.
(b) Suppose \( N \) is a non-trivial normal subgroup of \( \mathcal{F} \). Show that \( N = \mathcal{F} \).

7. Consider the group presented below.
\[ G = \langle a, b, c \mid a^2, b^2, c^3, aba^{-1}b^{-1}, cbcb, cac \rangle \]

(a) Prove that \( G \) has twelve elements.
(b) Find a normal form for \( G \).
(c) Is \( G \) isomorphic with \( D_6 \)? Explain your answer.

2 Groups with order divisible by a prime \( p \)

In this section, we will use group actions to discover some nice properties of a group \( G \), knowing only that the order of \( G \) is divisible by some prime \( p \).

Suppose \(|G| = n = p^k m\), where \( p \) is prime, and \( GCD(p, m) = 1 \).

Let \( S \) be the set of all subsets of \( G \) with order \( p^k \). We will let \( G \) act on \( S \) by left multiplication, that is, for any \( g \in G \) and \( \alpha \in S \), we define
\[ g \cdot \alpha = g \alpha = \{ gs \mid s \in \alpha \} \]

1. Show that the definition above actually defines an action of \( G \) on \( S \).

\[ \circ \circ \circ \]

Recall that there are \( \binom{m}{k} = \frac{m!}{(m-k)!k!} \) distinct subsets of \( k \) elements in a finite set of size \( m \). In particular,
\[ |S| = \frac{n!}{(n-p^k)!p^k!} \]

2. Show that \( p \) does not divide \(|S|\) (you will need to use the fact that \( GCD(m, p) = 1 \)).

\[ \circ \circ \circ \]

Recall that \( S \) is partitioned by the orbits in \( S \) under the action of \( G \), that is
\[ |S| = \sum_{\text{distinct orbits } \mathcal{O}} |\mathcal{O}| \]

3. Show that there is some orbit \( \mathcal{O}_* \) with \( p \nmid |\mathcal{O}_*| \).
4. Let $\alpha \in \mathcal{O}_a$, the orbit found in the previous question, and let $G_\alpha$ be the stabilizer of $\alpha$ under the action of $G$ on $S$. Prove that $p^k$ divides $|G_\alpha|$.

\[ \text{\textbullet\textbullet\textbullet} \]

For the next two questions, suppose $H \leq G$ and that $H$ stabilizes $\alpha$.

5. Let $H$ act on $G$ by left multiplication. Explain why $\alpha$ is a union of orbits of the action of $H$ on $G$.

6. Use a fact from our homework this semester (about orbit sizes) to conclude that $|H| \mid p^k$.

7. Explain why we now see that $|G_\alpha| = p^k$, where $G_\alpha$ is the group in Question 4 above.

\[ \text{\textbullet\textbullet\textbullet} \]

We have now shown the following theorem, which has a flavor similar to Cauchy’s theorem, and is a partial converse to Lagrange’s theorem.

**Theorem 1.** Suppose $G$ is a group of order $p^k m$ for some prime $p$ and integer $m$ with $\gcd(m, p) = 1$, then $G$ has a subgroup with order $p^k$. 