Solution to prelim 2, MATH3230

1. (a) \(\{f_n\}_{n=0}^{\infty}\) converges to \(f\) uniformly means that for all \(\epsilon > 0\), there exist \(N\) such that for all \(n \geq N\), \(|f_n(x) - f(x)| < \epsilon\) for all \(x \in [0,1]\).

(b) \(F\) satisfies a Lipschitz condition if there exists a number \(K > 0\) such that

\[|F(t, x) - F(t, y)| \leq K|x - y|\]

for all \(t \in [a,b]\) and \(x, y \in [c,d]\).

If \(F\) is continuously differentiable, then \(\frac{\partial F}{\partial y}\) is bounded on \([a,b] \times [c,b]\), since it is a compact set. By the Mean Value Theorem, \(F\) satisfies a Lipschitz condition with Lipschitz constant \(K\), where \(K\) is the maximum value of \(\left|\frac{\partial F}{\partial y}\right|\) on \([a,b] \times [c,b]\).

2. (a) By Gronwall's Inequality,

\[f(t) \leq 0 \cdot \exp\left(\int_0^t C\, dt\right) = 0.\]

(b) We have

\[y_1(t) = y_1(a) + \int_a^t F(t, y_1(t)) \, dt\]

and

\[y_2(t) = y_2(a) + \int_a^t F(t, y_2(t)) \, dt.\]

Since \(y_1(a) = y_2(a)\), we have

\[y_1(t) - y_2(t) = \int_a^t F(t, y_1(t)) - F(t, y_2(t)) \, dt\]

and hence

\[|y_1(t) - y_2(t)| \leq \int_a^t |F(t, y_1(t)) - F(t, y_2(t))| \, dt \leq \int_a^t C|y_1(t) - y_2(t)| \, dt = C \int_a^t |y_1(t) - y_2(t)| \, dt\]

(c) Applying part (a) to the result of part (b), we see that

\[|y_1(t) - y_2(t)| = 0\]

for all \(a < t < b\). This implies \(y_1(t) = y_2(t)\) for all \(a < t < b\).
3. (a) \[ \frac{\partial u}{\partial n} = \nabla u \cdot n. \]

(b) \[
\int_{S(r_0, \epsilon)} \frac{\partial u}{\partial n} \, d\sigma = \int_{S(r_0, \epsilon)} \nabla u \cdot n \, d\sigma \]
\[= \int_{B(r_0, \epsilon)} \nabla \cdot (\nabla u) \, dV \quad \text{(Gauss's Theorem)} \]
\[= \int_{B(r_0, \epsilon)} \Delta u \, dV \]
\[= 0 \]

(c) By representation theorem,
\[
u(r_0) = \frac{1}{4\pi} \int_{S(r_0, \epsilon)} \frac{1}{|r - r_0|} \frac{\partial u}{\partial n} - \frac{u}{|r - r_0|} \, d\sigma - \frac{1}{4\pi} \int_{B(r_0, \epsilon)} \nabla \frac{1}{|r - r_0|} \, dV \]
\[= \frac{1}{4\pi} \int_{S(r_0, \epsilon)} \frac{1}{\epsilon} \frac{\partial u}{\partial n} \, d\sigma - \frac{1}{4\pi} \int_{S(r_0, \epsilon)} \frac{u}{\epsilon^2} \, d\sigma - 0 \]
\[= \frac{1}{4\pi \epsilon^2} \int_{S(r_0, \epsilon)} u \, d\sigma \]

4. (a) For \( u(t, x) = f(t)g(x), u_x - u_{tt} = f(t)g'(x) - f''(t)g(x) \). Hence we need to solve
\[
f(t)g'(x) - f''(t)g(x) = 0 \]
\[
f(t)g'(x) = f''(t)g(x) \]
\[
\frac{g'(x)}{g(x)} = \frac{f''(t)}{f(t)} \]
Since the left hand side only depends on \( x \) and the right hand side only depends on \( t \), there is a constant \( \mu \) such that
\[
\frac{g'(x)}{g(x)} \equiv \mu \equiv \frac{f''(t)}{f(t)} \]

There are three cases:
- Case 1: \( \mu > 0 \). Then we have
\[
f(t) = Ae^{\sqrt{\mu}t} + Be^{-\sqrt{\mu}t} \]
and
\[
g(x) = Ce^{\mu x} \]
where \( A, B, C \) are arbitrary constants.
Case 2: \( \mu < 0 \). In this case we have
\[
f(t) = A \cos(\sqrt{-\mu} t) + B \cos(\sqrt{-\mu} t)
\]
and
\[
g(x) = C e^{\mu x}
\]
where \( A, B, C \) are arbitrary constants.

Case 3: \( \mu = 0 \). Then we have \( f''(t) = 0 = g'(x) \) and hence
\[
f(t) = A + B t
\]
and
\[
g(x) = C,
\]
where \( A, B, C \) are arbitrary constants.

(b) Only in case 2 above \( f \) is periodic but not constant. Therefore, \( g(x) = C e^{\mu x} \) with \( \mu \) being negative. Hence,
\[
\lim_{x \to \infty} g(x) = 0.
\]

5. The characteristic equations are
\[
\frac{dx}{ds} = x, \quad \frac{dy}{ds} = -y, \quad \frac{du}{ds} = 0.
\]
We have
\[
x(s) = x_0 e^s, \quad y(s) = y_0 e^{-s}.
\]
Since the initial conditions are only the line \( \{(x, x) : x > 0\} \), we have \( y_0 = x_0 \). Hence
\[
u(x(s), y(s)) = x_0.
\]
Solving for \( x_0 \) in terms of \( x(s) \) and \( y(s) \), we see \( x_0 = \sqrt{x(s)y(s)} \).
Hence, \( u(x, t) = \sqrt{xy} \) is a solution.