Solution to Assignment 9, MATH3230

1. The characteristic equations are
\[
\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = x, \quad \frac{du}{ds} = 0.
\]

We have
\[x(s) = s + x_0.\]

Since the condition on \(u\) is \(u(0, y) = f(y)\), we know \(x_0 = 0\) and
\[x(s) = s, \quad y(s) = s^2 + y_0, \quad u(s) = u_0.\]

The Characteristic curves are the curves \(y = \frac{s^2}{2} + y_0\), for \(y_0 \in \mathbb{R}\). Therefore, they are disjoint and cover the whole plane. Hence, the solution exists for all \(x, y \in \mathbb{R}\) and is unique.

We can compute
\[u(x, y) = u_0 = f(y_0) = f(y - \frac{s^2}{2} - x_0s) = f(y - \frac{x^2}{2}).\]

Since \(f\) is smooth, \(u\) is also smooth.

2. The characteristic equations are
\[
\frac{dx}{ds} = 1, \quad \frac{dy}{ds} = 2u, \quad \frac{du}{ds} = 0.
\]

We have
\[x(s) = s + x_0.\]

Since the condition on \(u\) is \(u(0, y) = y^2\), we know \(x_0 = 0\) and
\[x(s) = s, \quad y(s) = 2u_0s + y_0, \quad u(s) = u_0 = y_0^2.\]

Characteristic curves are the straight lines with slope equal to \(2y_0^2\) passing through \((0, y_0)\).

There will be intersecting characteristic curves (two lines with different slopes will always intersect). Solving the corresponding ODE on the two characteristic curves will give two different values of \(u\) at the intersection point because the two solution are both constant with respect to \(s\) and with different initial values. Hence, we do not have a solution \(u\) for all \(x, y\).

But actually we have local solutions near the line \(\{x = 0\}\). Recall that the characteristics are parametrized by
\[x(s) = s, \quad y(s) = 2y_0^2s + y_0.\]
Therefore, on the line $x = 0$,
\[
\begin{bmatrix}
\frac{\partial x}{\partial s} & \frac{\partial x}{\partial y} \\
\frac{\partial y}{\partial s} & \frac{\partial y}{\partial y_0}
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
2y_0^2 & 4y_0s + 1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
2y_0^2 & 1
\end{bmatrix}
\]
is non-singular. This shows that a local solution exists.

For fixed $(x, y)$, we want to find the corresponding $y_0$’s such that the characteristics curve from $(0, y_0)$ passes through $(x, y)$. So we are solving
\[
x = s, \quad y = 2y_0^2s + y_0.
\]
This gives
\[
y_0 = \frac{-1 \pm \sqrt{1 + 8xy}}{4x}
\]
as long as $1 + 8xy \geq 0$ (This also shows if $1 + 8xy < 0$, then there is no characteristic curve passing through $(x, y)$) and hence
\[
u(x, y) = \left(\frac{-1 \pm \sqrt{1 + 8xy}}{4x}\right)^2.
\]
Now we need to decide whether we need to choose the $+$ or $-$ sign (due to continuity we can only either have $+$ sign for all those $(x, y)$ or $-$ sign for all those $(x, y)$). But Observe that the function
\[
\left(\frac{-1 - \sqrt{1 + 8xy}}{4x}\right)^2
\]
blooms up when $x \to 0$. Therefore, we should choose
\[
u(x, y) = \left(\frac{-1 + \sqrt{1 + 8xy}}{4x}\right)^2.
\]
By either L’Hospital’s Rule or expanding its power series, we can check that $u(0, y) = y^2$ and thus $u$ is the local solution of the PDE, defined on the open set $\{1 + 8xy > 0\}$.

3. The characteristic equations are
\[
\frac{dx}{ds} = x, \quad \frac{dy}{ds} = 2y, \quad \frac{du}{ds} = 0.
\]
We have
\[
x(s) = x_0e^s.
\]
Since the condition on $u$ is $u(1, y) = f(y)$, we know $x_0 = 1$ and
\[
x(s) = e^s, \quad y(s) = y_0e^{2s}, \quad u(s) = u_0 = f(y_0).
\]
The Characteristic curves are the curves \( y = y_0 x^2 \) with \( x \in (0, \infty) \) (because \( x = e^s \)), and for \( y_0 \in \mathbb{R} \). Therefore, they are disjoint but do not cover the whole plane.

We can compute, for \( x \in (0, \infty) \),

\[
    u(x, y) = f(y_0) = f\left(\frac{y}{x^2}\right).
\]

This function is not extendable to the whole plan (For instance, in general it does not have a limit at \((0, 0)\)). Therefore, there will not be a solution \( u \) for all values of \( x, y \).

But if we restrict our domain to the open right half plane \( \{(x, y) : x > 0\} \), the global solution in this domain exists and is unique. It is also smooth because we have computed \( u(x, y) = f\left(\frac{y}{x^2}\right) \) and \( f \) is smooth.

4. The characteristic equations are

\[
\frac{dx}{ds} = u, \quad \frac{dy}{ds} = 1, \quad \frac{du}{ds} = 1.
\]

We have

\[
u(s) = u_0 + s
\]

and hence

\[
x(s) = \frac{s^2}{2} + u_0 s + x_0, \quad y(s) = s + y_0, \quad u(s) = u_0 + s.
\]

Since the curve (from now we call it \( C \))

\[
x(t) = t^2, \quad y(t) = 2t
\]

is given by the equation

\[
x = \frac{y^2}{4},
\]

we know

\[
x_0 = \frac{y_0^2}{4}, \quad u_0 = \frac{y_0}{2}.
\]

Therefore, we can write

\[
x(s) = \frac{s^2}{2} + \frac{y_0 s}{2} + \frac{y_0^2}{4}, \quad y(s) = s + y_0, \quad u(s) = \frac{y_0}{2} + s.
\]

Observe that

\[
x(s) = \frac{y(s)^2}{4} + \frac{s^2}{4}.
\]

Hence the characteristic curves are parabolas (opening towards right) sitting on the right hand side of \( C \), touching \( C \) at the point \((x_0, y_0)\). Hence,
any two characteristic curves will intersect each other. The characteristic curves cover the area \{ (x, y) : 4x - y^2 \geq 0 \}.

When we solve for \( u \) using the above equations, we see

\[ u = \frac{y_0}{2} + s, \quad 4x - y^2 = s^2. \]

There are two case.

When \( s > 0 \), we see

\[ s = \sqrt{4x - y^2} \]

and hence

\[ y = y_0 + \sqrt{4x - y^2}. \]

Then we can conclude that

\[ u = \frac{y_0}{2} + s = \frac{y - \sqrt{4x - y^2}}{2} + \sqrt{4x - y^2} = \frac{y + \sqrt{4x - y^2}}{2}. \]

We can check that this is a function which is smooth in the interior of \{ (x, y) : 4x - y^2 \geq 0 \} and continues on \{ (x, y) : 4x - y^2 \geq 0 \}, which satisfies the PDE in the interior.

Similarly, \( u = \frac{y - \sqrt{4x - y^2}}{2} \) is also a solution with this property.

Therefore, when we restrict the domain to \{ (x, y) : 4x - y^2 \geq 0 \}, we have non-unique solutions.

These solutions cannot be extended smoothly beyond the boundary, as its gradient goes to infinity at the boundary.

This phenomenon is caused by the fact that the jacobian

\[
\begin{bmatrix}
\frac{\partial x}{\partial s} & \frac{\partial x}{\partial y} \\
\frac{\partial x}{\partial s} & \frac{\partial x}{\partial y_0}
\end{bmatrix}
\]

is singular at the boundary.