Solution to Assignment 6, MATH3230

1. \( E = \frac{\omega(t)^2}{2} + 1 - \cos(\theta(t)) \). \( E \) is at least 0 because \( \omega^2 \geq 0 \) and \( 1 - \cos(\theta) \geq 0 \) for any value of \( \omega \) and \( \theta \).

\[
\frac{dE}{dt} = \omega(t)\omega'(t) + \sin(\theta(t))\theta'(t)
= \omega(t)(-\omega(t) - \sin(\theta(t))) + \sin(\theta(t))\omega(t)
= -\omega^2(t)
\leq 0.
\]

Therefore, \( E \) is decreasing. By Monotone Convergence Theorem, \( \lim_{t \to \infty} E(t) \) exists.

2. Applying challenge problems I and IV to the function \( f(t) = g(y(t)) \),

\[
\lim_{t \to \infty} \frac{d}{dt}g(y(t)) = 0.
\]

By chain rule,

\[
\frac{d}{dt}g(y(t)) = \nabla g(y(t)) \cdot y'(t) = \nabla g(y(t)) \cdot F(y(t))
\]

and result follows.

3. Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be

\[
F(\theta, \omega) = (\omega, -\omega - \sin(\theta))
\]

and \( g : \mathbb{R}^2 \to \mathbb{R} \) be

\[
g(\theta, \omega) = E(\theta, \omega) = \frac{\omega^2}{2} + 1 - \cos \theta.
\]

Then \( F \) and \( g \) are continuously differentiable because they are compositions of elementary functions.

Let \( y(t) = (\omega(t), \theta(t)) \). Then \( y(t) \) solves

\[
y'(t) = F(y(t)).
\]

By challenge problem II, \( y(t) \) is bounded.

By problem 1, \( \lim_{t \to \infty} g(y(t)) \) exists.

Therefore, we have checked all the hypothesis of problem 2 are satisfied and we can apply problem 2 to get

\[
\lim_{t \to \infty} \nabla g(y(t)) \cdot F(y(t)) = 0.
\]
We can calculate
\[ \nabla g = \left( \frac{\partial g}{\partial \theta}, \frac{\partial g}{\partial \omega} \right) = (\sin \theta, \omega) \]
and
\[ \nabla g \cdot F = (\sin \theta, \omega) \cdot (\omega, -\omega - \sin(\theta)) = -\omega^2 \]

Therefore, we have
\[ \lim_{t \to \infty} -\omega(t)^2 = 0 \]
and hence
\[ \lim_{t \to \infty} \omega(t) = 0. \]

4. By problem 2, \( \lim_{t \to \infty} \omega(t) \) exists. By problem 3 with \( g(\theta, \omega) = \omega \) and \( F(\theta, \omega) = (\omega, -\omega - \sin(\theta)) \) we have that
\[ 0 = \lim_{t \to \infty} \nabla g \cdot F(\theta(t), \omega(t)) = \lim_{t \to \infty} \langle 0, 1 \rangle \cdot \langle \omega(t), -\omega(t) - \sin(\theta(t)) \rangle = \lim_{t \to \infty} -\sin(\theta(t)). \]