Solution to Assignment 5, MATH3230

1. Because $f > 0$, we see $x(t)$ is a strictly increasing function of $t$. Therefore, its inverse exists and $\frac{dt}{dx} = \left(\frac{dx}{dt}\right)^{-1}$. Therefore,

$$\frac{1}{g(u)} \leq \frac{dt}{dx}(u) \leq \frac{1}{f(u)}$$

For $t > t_0$,

$$t - t_0 = \int_{x(t_0)}^{x(t)} \frac{dt}{dx}(u)du$$

and thus

$$\int_{x(t_0)}^{x(t)} \frac{1}{g(u)}du \leq t - t_0 \leq \int_{x(t_0)}^{x(t)} \frac{1}{f(u)}du$$

2. Suppose $J = (a, b)$ is a bounded open interval and $\theta, \omega$ are functions on $J$ satisfying the ODE. We first want to show that $\omega$ is bounded on $J$:

Method 1: First, suppose $I = (c, d) \subseteq J$ is a subinterval such that $\omega(t) < -2$ for all $t \in I$. Then we have, for all $t \in I$,

$$\omega'(t) = -\omega(t) - \sin(\theta(t)) \in [-\omega(t) - 1, -\omega(t) + 1]$$

Therefore,

$$0 \leq \omega'(t) \leq -\omega(t) + 1$$

Since the function $-\omega(t) + 1$ is positive on $I$, we can apply problem 1 and get, for all $s, t \in I$ with $s > t$,

$$\int_{\omega(t)}^{\omega(s)} \frac{1}{-u + 1}du \leq s - t$$

$$[- \log(-u + 1)]_{\omega(t)}^{\omega(s)} \leq s - t$$

$$\log\left(\frac{1 - \omega(t)}{1 - \omega(s)}\right) \leq s - t$$

$$\frac{1 - \omega(t)}{1 - \omega(s)} \leq e^{s-t} \leq e^{b-a}$$

Also note that $\omega(t) < \omega(s) < -2$ and thus $1 - \omega(t) > 1 - \omega(s) > 0$. Therefore, we actually have concluded that, for $s$ and $t$ lying in a subinterval where $\omega < -2$,

$$e^{-(b-a)} \leq \left|\frac{1 - \omega(t)}{1 - \omega(s)}\right| \leq e^{b-a}$$
Similarly, by either considering $\tilde{\omega}(t) = \omega(-t)$ or using a $0 > g(s) > x'(t) > f(x)$ version of problem 1, we can show that for $s$ and $t$ lying in a subinterval where $\omega > 2$, we also have

$$e^{-(b-a)} \leq \frac{|1 - \omega(t)|}{1 - \omega(s)} \leq e^{b-a}$$

Now we are almost done. There are two cases

- **Case 1:** $\omega < -2$ or $\omega > 2$ on the whole $J$. In this case we can just take $I = J$ in the above argument and we have

$$e^{-(b-a)} \leq \frac{|1 - \omega(t)|}{1 - \omega(s)} \leq e^{b-a}$$

for all $s, t \in J$. Hence $\omega$ is bounded on $J$.

- **Case 2:** there is some value of $\omega$ on $J$ lying in $[-2, 2]$. Then, for each $I$ above, we can extend it so that at one end of it, $\omega$ has value $-2$ or $2$. Taking $s$ equal to this endpoint in the above argument, we see

$$|1 - \omega(t)| \leq e^{b-a}$$

for all $t \in I$. But $I$ is an arbitrary interval in $J$ where $|\omega| > 2$. Therefore, we can conclude that for all $t \in J$ such that $|\omega(t)| > 2$, $|1 - \omega(t)| \leq e^{b-a}$. And therefore $\omega$ is bounded on $J$.

**Method 2:** Fix $s \in J$. For all $t \in J$ such that $t > s$,

$$\omega(t) = \omega(s) + \int_s^t \omega'(u)du$$

$$\omega(t) = \omega(s) + \int_s^t -\omega(u) - \sin(\theta(u))du$$

$$|\omega(t)| \leq |\omega(s)| + \int_s^t |\omega(u)|du + \int_s^t |\sin(\theta(u))|du$$

$$\leq |\omega(s)| + \int_s^t |\omega(u)|du + \int_s^t 1du$$

$$\leq |\omega(s)| + \int_s^t |\omega(u)|du + t - s$$

$$\leq (|\omega(s)| + (b - a)) + \int_s^t |\omega(u)|du$$

Therefore, by Gronwall’s Inequality, for $t > s$

$$|\omega(t)| \leq (|\omega(s)| + (b - a))e^{\int_s^t 1du} \leq (|\omega(s)| + (b - a))e^{b-a}$$
Similarly, by considering $\tilde{\omega}(t) = \omega(-t)$, we can see that for all $t < s$,

$$|\omega(t)| \leq (|\omega(s)| + (b - a))e^{b-a}$$

Thus $|\omega|$ is bounded by $(|\omega(s)| + (b - a))e^{b-a}$ on $J$ and we are done.

Now we are left with the boundedness of $\theta$. Let $K$ be an upper bound of $|\omega|$ on $J$. Fix some $s \in J$. Since

$$|\theta'(t)| = |\omega(t)| \leq K$$

for all $t$, we see that

$$|\theta(t)| \leq |\theta(s)| + K|t - s| \leq |\theta(s)| + K(b - a)$$

Therefore, $\theta$ is bounded on $J$.

3. The autonomous system is given by

$$\frac{d}{dt}(\theta, \omega) = F(\theta, \omega) = (\omega, -\omega - \sin \theta)$$

The equilibria are all the points $(\theta, \omega)$ such that $(\omega, -\omega - \sin \theta) = (0, 0)$, which are all the points $(\theta, \omega) = (n\pi, 0)$, where $n$ is any integer.

The Jacobian of $F$ is given by

$$\left[ \begin{array}{cc} \frac{\partial}{\partial \theta} \omega & \frac{\partial}{\partial \omega} \omega \\ -\cos \theta & -1 \end{array} \right] = \left[ \begin{array}{cc} 0 & 1 \\ -1 & -1 \end{array} \right], \ n \text{ is even}$$

$$\left[ \begin{array}{cc} \frac{\partial}{\partial \theta} \omega & \frac{\partial}{\partial \omega} \omega \\ -\cos \theta & -1 \end{array} \right] = \left[ \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right], \ n \text{ is odd}$$

When $n$ is even, $\left[ \begin{array}{cc} 0 & 1 \\ -1 & -1 \end{array} \right]$ has eigenvalues $-1 \pm \sqrt{3} \frac{i}{2}$, which have negative real parts. Therefore, it is an asymptotically stable equilibrium.

When $n$ is odd, $\left[ \begin{array}{cc} 0 & 1 \\ 1 & -1 \end{array} \right]$ has eigenvalues $\frac{-1 \pm \sqrt{5}}{2}$. One of them is positive and the other is negative. Therefore, it is an unstable equilibrium, which is also a saddle point.

Remark: The above result agrees with our intuition: for even $n$ the equilibrium corresponds to the situation where the mass is below the support. For odd $n$ the equilibrium corresponds to when the mass is resting right above the support. A little push to the mass will make it fall down due to gravity.
4. Fix $\omega_0 > 1$.

Existence of $t_0$: Since $\lim_{t \to -\infty} \omega(t) = \infty$, there is $t_1$ such that $\omega(t) > \omega_0$ for all $t \leq t_1$.

If there is also $t_0$ such that $\omega(t_0) < \omega_0$, then by intermediate value theorem, there is $t \in (t_1, t_0)$ such that $\omega(t) = \omega_0$. Therefore, we just need to show there exists $t_0$ such that $\omega(t_0) < \omega_0$.

Suppose there is no such $t_0$. Then $\omega(t) > \omega_0$ for all $t \in \mathbb{R}$. And Thus

$$\omega'(t) = -\omega(t) - \sin(\theta(t)) < -\omega_0 + 1 = - (\omega_0 - 1)$$

Therefore, by mean value theorem, for all $t > t_1$,

$$\omega(t) < \omega(t_1) - (t - t_1) (\omega_0 - 1)$$

but the quantity on the right hand side goes to $-\infty$ as $t \to \infty$ (recall $\omega_0 > 1$). And this is a contradiction since we have assumed $\omega(t) > \omega_0$ for all $t$.

Uniqueness of $t_0$: Again, if $\omega(t) > 1$, then

$$\omega'(t) = -\omega(t) - \sin(\theta(t)) < -1 + 1 = 0 \quad (1)$$

Suppose $s \in \mathbb{R}$ such that $\omega(s) > 1$. Then as above we have $\omega'(s) < 0$. Therefore, there is $\epsilon > 0$ such that $\omega(s)$ is strictly larger that $\omega(s)$ on $(s - \epsilon, s)$. Now observe that $\omega(t) > \omega(s)$ for all $t < s$. For otherwise let $u$ be maximum such that $u < s$ and $\omega(u) \leq \omega(s)$ (exists because the set $\{t : \omega(t) \leq \omega(s)\}$ is closed). Then $\omega(u) = \omega(s)$. By mean value theorem, there is $v \in (u, s)$ such that

$$\omega'(v) = \frac{\omega(s) - \omega(u)}{s - u} = 0 \quad (2)$$

But $u < v < s$ implies $\omega(v) > \omega(s) > 1$. This with (2) contradicts (1).

In conclusion, we have proved that for $\omega(s) > 1$, $\omega(t) > \omega(s)$ for all $t < s$.

Now suppose there is another $s \neq t_0$ such that $\omega(s) = \omega(t_0) = \omega_0 > 1$, then apply the above observation to the larger number among $s$ and $t_0$, we get a contradiction.