1. The relationship is given by \( V_0 = iR \) and \( V_1 - V_0 = L \frac{di}{dt} \). Differentiating the first equation and substitute into the second, we get
\[
V_1 - V_0 = \frac{L}{R} \frac{dV_0}{dt}
\]
If \( V_1 \) is set to be 0, we get
\[
-V_0 = \frac{L}{R} \frac{dV_0}{dt}
\]
\[
d\frac{dV_0}{dt} = -\frac{R}{L} V_0
\]
\[
V_0 = Ce^{-\frac{R}{L} t}
\]
(By the polynomial method)

Plugging in \( t = 0 \) and \( V_0 = v_0 \), we see \( C = v_0 \). Therefore, \( V_0 = v_0 e^{-\frac{R}{L} t} \).

2. Square the inequalities, what we need to prove becomes
\[
\sum_{i=1}^{n} x_i^2 \leq \left( \sum_{i=1}^{n} |x_i| \right)^2 \leq n \sum_{i=1}^{n} x_i^2
\]
Expanding the middle expression, it becomes
\[
\sum_{i=1}^{n} x_i^2 \leq \sum_{i=1}^{n} |x_i|^2 + \sum_{i \neq j} |x_i||x_j| \leq n \sum_{i=1}^{n} x_i^2
\]
Since \( \sum_{i \neq j} |x_i||x_j| \geq 0 \), the left inequality is clear. Thus, it remains to prove
\[
\sum_{i=1}^{n} |x_i|^2 + \sum_{i \neq j} |x_i||x_j| \leq n \sum_{i=1}^{n} x_i^2
\]
which is equivalent to
\[
\sum_{i \neq j} |x_i||x_j| \leq (n - 1) \sum_{i=1}^{n} x_i^2
\]
To prove this, we proceed as follow:
\[
\sum_{i \neq j} |x_i||x_j| \leq \frac{1}{2} \left( \sum_{i \neq j} x_i^2 + x_j^2 \right) = \frac{1}{2} \left( \sum_{i \neq j} x_i^2 + \sum_{i \neq j} x_j^2 \right)
\]
\[
= \frac{1}{2} \left( (n - 1)(\sum_{i} x_i^2) + (n - 1)(\sum_{j} x_j^2) \right) = (n - 1) \sum_{i=1}^{n} x_i^2
\]
3. \[ |AB| = \sum_{i,j} |(AB)_{ij}| \]
   \[ = \sum_{i,j} | \sum_k A_{ik} B_{kj} | \]
   \[ = \sum_{i,j,k} |A_{ik} B_{kj}| \]
   \[ \leq \sum_{i,j,k,l} |A_{ik} B_{lj}| \]
   \[ = (\sum_{i,k} |A_{ik}|)(\sum_{l,j} |B_{lj}|) \]
   \[ = |A||B| \] (This involves the expression above and some extra terms!)

4. There is a \( n \times m \) matrix \( A \) such that \( Tx = Ax \) for all \( x \in \mathbb{R}^m \). Let \( M = \max_{i,j} |A_{ij}| \). For all \( x \in \mathbb{R}^m \),

\[ |Tx| = \sum_i |(Tx)_i| \]
\[ = \sum_i | \sum_j A_{ij} x_j | \]
\[ \leq \sum_i \sum_j |A_{ij}| |x_j| \]
\[ \leq \sum_i \sum_j M |x_j| \]
\[ = nM \sum_j |x_j| \]
\[ = nM |x| \]

Challenge problems:

1. Every linear transformation \( T \) is given by matrix multiplication. Therefore, each coordinate of \( Tx \) is a polynomial (more precisely, linear combination) of the \( x_i \)'s. Therefore, every coordinate of \( Tx \) is a continuous function of \( x \) and hence \( Tx \) is a continuous function of \( x \).

2. For each fixed pair of \( i,j \), we have \( |a_{ij}^{(k)}| \leq |A^{(k)}| \). Therefore,

\[ \sum_{k=0}^{\infty} |a_{ij}^{(k)}| \leq \sum_{k=0}^{\infty} |A^{(k)}| < \infty \]
This means \( \sum_{k=0}^{\infty} a_{ij}^{(k)} \) converges absolutely.

For the second part, let \( A^{(n)} = \frac{A^n}{n!} \). Then

\[
|A^{(n)}| = \frac{|A^n|}{n!} \leq \frac{|A|^n}{n!} \quad \text{(by problem 3)}
\]

It is well known that \( \sum_{n=0}^{\infty} \frac{r^n}{n!} < \infty \) for every real number \( r \). Therefore, we see, for all \( A \),

\[
\sum_{n=0}^{\infty} |A^{(n)}| < \infty
\]

By the first part of this problem, we see that \( \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} A^{(n)} \) converges.

3. From the previous HW we have

\[
v(t) = \frac{gm}{\alpha} - \frac{gm}{\alpha} e^{-\frac{\alpha}{m}t} = \frac{gm}{\alpha} (1 - e^{-\frac{\alpha}{m}t}).
\]

\[
v(t) = \frac{gm}{\alpha} - \frac{gm}{\alpha} e^{-\frac{\alpha}{m}t} \\
= \frac{gm}{\alpha} (1 - e^{-\frac{\alpha}{m}t}) \\
= \frac{gm}{\alpha} \left( 1 - \left(1 + \left(-\frac{\alpha}{m}t\right) + \frac{1}{2!}\left(-\frac{\alpha}{m}t\right)^2 + \frac{1}{3!}\left(-\frac{\alpha}{m}t\right)^3 + \cdots \right) \right) \\
= \frac{gm}{\alpha} \left( \frac{\alpha}{m} t - \frac{1}{2!}\left(-\frac{\alpha}{m}t\right)^2 - \frac{1}{3!}\left(-\frac{\alpha}{m}t\right)^3 - \cdots \right) \\
= gt - \frac{1}{2!}g\left(-\frac{\alpha}{m}\right)t^2 - \frac{1}{3!}g\left(-\frac{\alpha}{m}\right)^2t^3 - \cdots \\
\to gt \text{ as } \alpha \to 0
\]

This limit is the same as the velocity if there is no air resistance.

(Remark: Another way to calculate the limit is to use the L’Hospital’s Rule.)