Solution to Assignment 1, MATH3230

1. For \( y = Ce^{\alpha x} \), \( y^{(i)} = C\alpha^i e^{\alpha x} \).
   Therefore, \( \sum_{i=0}^{n} a_i y^{(i)} = C \left( \sum_{i=0}^{n} a_i \alpha^i \right) e^{\alpha x} = 0 \).

2. The system is given by \( \frac{dY}{dt} = \alpha YM \) and \( \frac{dM}{dt} = \beta Y \).
   In reality, \( M, Y, \alpha \) are positive and \( \beta \) is negative.

3. The equation is given by \( \frac{dv}{dt} = g - \frac{\alpha v}{m} \).
   \[
   \int \frac{dv}{mg - \alpha v} = \frac{1}{m} \int dt \quad \text{assuming } mg - \alpha v \neq 0
   \]
   \[
   \frac{-1}{\alpha} \log |mg - \alpha v| = \frac{t}{m} + C
   \]
   \[
   mg - \alpha v = e^{\frac{\alpha t}{m} + C}
   \]
   \[
   v = mg - \frac{\alpha}{m} \left( e^C \right) e^{\frac{-\alpha t}{m}}
   \]
   \[
   \therefore \quad v = \frac{gm}{\alpha} + Be^{-\frac{\alpha t}{m}}
   \]
   where \( B \) is any nonzero constant. Note that we have assumed \( mg - \alpha v \neq 0 \).
   We can also observe that \( v \equiv \frac{mg}{\alpha} \) is also a solution. Therefore, by the uniqueness of solution, if \( mg - \alpha v = 0 \) at some point of a solution, that solution will be \( v \equiv \frac{mg}{\alpha} \).
   To conclude, \( v = \frac{2m}{\alpha} + Be^{-\frac{\alpha t}{m}} \), where \( B \) is any real constant.

4. Let \( v = \) velocity of the mass = \( L \frac{d\theta}{dt} \).
   So we have \( m \frac{dv}{dt} = -mg \sin \theta - \alpha v \).
   If we approximate \( \sin \theta \) by \( \theta \) when \( |\theta| \) is small, we have
   \[
   mL \frac{d^2\theta}{dt^2} = -mg \theta - \alpha L \frac{d\theta}{dt}
   \]
   \[
   \therefore \quad \frac{d^2\theta}{dt^2} + \frac{\alpha}{m} \frac{d\theta}{dt} + \frac{g}{L} \theta = 0
   \]
   We obtain a second order ODE with constant coefficients, with \( x^2 + \frac{\alpha}{m} x + \frac{g}{L} = 0 \) as characteristic equation. Therefore, there are 3 cases:
   
   • \( \left( \frac{\alpha}{m} \right)^2 - 4 \frac{g}{L} > 0 \):
In this case we have \( \theta = A \exp\left(\frac{-(\pm) + \sqrt{(\pm)^2 - 4gL}}{2} t\right) + B \exp\left(\frac{-(\pm) - \sqrt{(\pm)^2 - 4gL}}{2} t\right) \)
as general solutions for the differential equation, where \( A \) and \( B \) are arbitrary constants.

• \( (\frac{\alpha}{m})^2 - 4gL = 0: \)
  This means \( \frac{\alpha}{m} = 2 \frac{\sqrt{g}}{L} \) and the root to the characteristic equation is
  \( -\frac{\sqrt{g}}{L} \). We have \( \theta = A \exp\left(-\frac{\sqrt{g}}{L} t\right) + B t \exp\left(-\frac{\sqrt{g}}{L} t\right) \), where \( A \) and \( B \) are arbitrary constants.

• \( (\frac{\alpha}{m})^2 - 4gL < 0: \)
  In this case we have \( \theta = A \exp\left(-\frac{\alpha}{2m} t\right) \sin\left(\frac{\sqrt{4gL - (\pm)^2}}{2} t\right) + B \exp\left(-\frac{\alpha}{2m} t\right) \cos\left(\frac{\sqrt{4gL - (\pm)^2}}{2} t\right) \)
as general solutions for the differential equation, where \( A \) and \( B \) are arbitrary constants.