Recall that a function \( f : X \to Y \) is injective if for all \( x_1, x_2 \in X \), \( f(x_1) = f(x_2) \) implies \( x_1 = x_2 \), and \( f \) is surjective if for all \( y \in Y \), there is an \( x \in X \) such that \( f(x) = y \). A function \( g : Y \to X \) is a right inverse for \( f \) if for all \( y \in Y \), \( f(g(y)) = y \), and \( g \) is a left inverse for \( f \) if for all \( x \in X \), \( g(f(x)) = x \).

**Proposition 1.** A function \( f : X \to Y \) is injective if and only if there is a function \( g : Y \to X \) that is a left inverse for \( f \) and any such right inverse \( g \) is surjective.

**Proposition 2.** A function \( f : X \to Y \) is surjective if and only if there is a function \( g : Y \to X \) that is a right inverse for \( f \) and any such left inverse \( g \) is injective.

Note that Proposition 2 is relatively natural, while Proposition 1 needs the axiom of choice from set theory. We say that the cardinality of a set \( X \) is less than or equal to the cardinality of a set \( Y \) and we write \( \#X \leq \#Y \) if there is a function \( f : X \to Y \) which is injective, or equivalently there is a function \( g : Y \to X \) which is surjective. For any two sets \( X, Y \), it turns out that \( \#X \leq \#Y \) or \( \#Y \leq \#X \).

We say that two sets have the same cardinality and \( \#X = \#Y \) if there are functions \( f : X \to Y \), \( g : Y \to X \) that are both left and right inverses of each other. The following is the Shroeder-Bernstein Theorem.

**Theorem 3.** For any two sets \( X, Y \), if \( \#X \leq \#Y \) and \( \#Y \leq \#X \), then \( \#X = \#Y \).

Recall that for any set \( X \), the power set of \( X \) is \( \mathcal{P}(X) = \{ A \subset X \} \), the set of subsets of \( X \).

**Theorem 4** (Cantor, Theorem 3.7 in the text). For any set \( X \), \( \#X < \#\mathcal{P}(X) \). In other words, the power set \( \mathcal{P}(X) \) has a strictly larger cardinality than \( X \).

**Proof.** Clearly \( \#X \leq \#\mathcal{P}(X) \), since the function \( f(x) = \{ x \} \) is injective. To show that the cardinalities are strict, suppose that \( f : X \to \mathcal{P}(X) \) is surjective. Then define the set

\[
X_0 = \{ x \in X \mid x \notin f(x) \} \subset X.  
\]

It is easy to check that \( X_0 \in \mathcal{P}(X) \) is not in the image of \( f \), contradicting surjectivity of \( f \). \( \square \)
Define \( \#\mathbb{N} = \aleph_0 \), \( \#\mathbb{R} = \aleph_1 \), and \( \#\mathcal{P}(\mathbb{R}) = \aleph_2 \), where \( \mathbb{N} = \{1, 2, 3, \ldots \} \) are the natural numbers.

**Problems**

1. Prove \( \aleph_0 < \aleph_1 < \aleph_2 \).

2. Define the function \( f : \mathbb{N} \to \mathcal{P}(\mathbb{N}) \), by \( f(n) = \{n + 1, 2\lfloor n/2 \rfloor\} \in \mathcal{P}(\mathbb{N}) \), where \( \lfloor \rfloor \) is the floor function defined previously. What is the set defined by (1)?

3. Problem 3.21 in the text.

4. Find the cardinality of the set of all infinite sequences \( \{(n_1, n_2, \ldots, n_k, \ldots) \mid n_k \in \mathbb{N}\} \) of elements of \( \mathbb{N} \). As much of a proof that you can provide is appreciated.

5. Find the cardinality of the set of all finite sequences \( \{(n_1, n_2, \ldots, n_k) \mid n_k \in \mathbb{N}\} \) of elements of \( \mathbb{N} \). As much of a proof that you can provide is appreciated.