The Predicate Calculus

1. Introduction

The predicate calculus is an extension of the propositional calculus that includes the notion of quantification. Instead of dealing only with statements, which have a definite truth-value, we deal with the more general notion of predicates, which are assertions in which variables appear. These variables are presented as “ranging” over some given sets, and quantification is a process that applies to these variables. Statements are here viewed as special predicates in which there are either no variables at all or in which all variables have been quantified. All of the operations of the propositional calculus extend to predicates virtually without change. What needs to be understood, however, is how these operations interact with quantification.

We shall begin by giving a few simple examples of variables and quantification and then turn to describing what we mean by these in general.

2. Some examples

Variables occur in all parts of mathematics, starting with basic algebra, set theory, number theory, and calculus.

For example, the concept of a polynomial includes the notion of a variable. When we write $x^3 + 17x + 10$, we think of $x$ as a variable ranging over some set of numbers, say the
real numbers. The polynomial itself is a recipe for performing certain algebraic operations on the variable $x$.

In set theory, we often consider two sets $S$ and $T$ and then, in some manner, define a function $f : S \to T$. We denote the function values by $f(s)$, where $s$ denotes a variable ranging over $S$. Sometimes, we may wish to write $t = f(s)$ and call $s$ the independent variable and $t$ the dependent variable.

In calculus, we frequently do what the preceding paragraph just described. The notion of a variable is particularly well-suited to calculus, since it studies the relative rates of different variables, integration with respect to a variable, etc.

In these areas, we often make statements in which we refer to a certain set of values that the variable ranges over. For example, we may say that the polynomial expression $x^2 + 1$ assumes positive values for all values of the real variable $x$. Or we may say that $\cos^2(x) \leq 1$ for all real $x$. Alternatively, we may wish to say that a certain equation has a solution, as when we say "The polynomial equation $x^3 + 17x + 10 = 0$ has a real root." This may be expressed as "There exists an $x$ such that $x^3 + 17x + 10 = 0,"$ provided it is understood that $x$ ranges over the real numbers.

These two ways of qualifying the range of values that the variable is permitted to attain—namely, for all and there exists — are known as quantification: the first is so-called universal quantification, the second existential quantification.

We now make some of these ideas more precise.

### 3. General elements of sets.

We shall assume the ideas from set theory described in the earlier chapter Set Theory.

No matter how a set $S$ is defined or presented, there is a notion of a general element of $S$. This is conceived as an object about which we know nothing other than it belongs to $S$. It has all of the properties needed to qualify as a member of $S$ but nothing further. Put another way, a general element of $S$ has exactly those properties shared by all elements of $S$.

For example, if $S$ is the set of all positive reals, then all we know about a general element of $S$ is that it is real and $> 0$. So, $3$, $\sqrt{2}$, and $\pi$ are positive reals, but they are not general elements of $S$ because we know more about them than that they are real and $> 0$.

Clearly the notion of a general element of a set $S$ is an abstract concept, since any concrete instance of an element in $S$ has something particular about it which is what enables us to single it out. The whole point of the concept is to enable us to reason conveniently about an element of $S$ qua element of $S$, without inadvertently introducing additional restrictive identifiers.

Sometimes mathematicians use the term arbitrary element, as a synonym for general element. For example, we may say “Let $x$ be an arbitrary element of the set $S$.” It is a common error for students to misunderstand this term and to then select a specific element of $S$ rather than to attend to a general one.

**Exercise 1.** (a) How would you describe a general element of the set of lower-case alphabetic characters $\{a, b, c, d\}$?

(b) Is $b$ a general element of the set $\{b\}$?

### 4. Variables and constants

The notion of a variable in logic or in mathematics is a linguistic construct. A variable ranging over a set $S$ is a symbol that represents a general element of $S$. Sometimes we
want to consider several variables ranging over \( S \) (or over various sets), and for this we use separate symbols for each. You are free to use whatever symbols you like for variables, provided your usage is consistent, both within itself and with respect to other sources you may be referencing. It is common to use lower case letters near the end of the alphabet for variables—such as \( r, s, t, u, v, w, x, y, z \)—but this is not mandatory. When many variables are used, their symbols often contain subscripts, as in \( x_1, x_2, x_3, y_1, y_2, \) etc.

The notion of a constant in \( S \) is closely analogous to that of a variable. It is also a linguistic construct. But in this case it is a symbol used to represent a particular element of \( S \), which is assumed to be held fixed for the duration of the discussion about \( S \), or for part of the discussion. Often letters from the early part of the alphabet are used for constants, such as \( a, b, k, B, C \), etc., but again this is not mandatory.

Sometimes notation for some constant spreads from one practitioner to another and attains wide currency, even universal acceptance. This occurs when the mathematics in question is especially important and/or the originators of the usage are very influential. In this way, symbols such as “0”, “1”, “2”, “\( \pi \)”, “\( e \)”, etc., have come to represent constants whose meaning is (nearly) universally understood.

5. Expressions

Variables and constants generally appear in larger linguistic constructs, the precise nature of which depends on the mathematical system that is being considered. Usually, such a system involves elements in one or more sets, various relations among the elements of the sets, operations on the elements of the sets, and possibly also various kinds of functions relevant to the system. Some particular elements, functions, operations, or relations may be singled out and denoted by special symbols, whereas others remain general. As we have discussed in Set Theory, it is important to assume that all of the sets considered in the mathematical system are subsets of some universal set \( U \).

Given a system with such elementary ingredients, we select symbols for representing elements, functions and operations (leaving relational symbols until later) and combine these symbols according to certain formation rules to obtain more and more complex strings of symbols that represent more complex objects of the system. Such well-formed strings are called expressions.

For an example of this, if one is studying integer arithmetic, one would consider the operations \(+, -, \) and \( \times \), as well as, perhaps, the related operations of raising to the \( n^{th} \) power, for non-negative integers \( n \). One would also want to consider the usual constants \( 0, \pm 1, \pm 2, \ldots \) etc., as mentioned above. Thus, one would expect expressions like \( x + y, 7x + 1, 3x^2 - 4x + 2, x^n - y^n, (x_1 + x_2 + x_3)(x_1 - x_2 + x_3) \), etc. Usually, parentheses are included among the allowed symbols in order to make the formation rule clear.

Exercise 2. Which of the following concatenations of symbols are valid expressions in integer arithmetic? (The concatenation does not include blanks or semicolons.) (a)1 ; (b) ((1) + (1)) ; (c) 1 + \times 2 ; (d)\( x\sqrt{x} \); (e) \( x^{2} \);

If one is dealing with real numbers, then one would want to include all the expressions of integer arithmetic (now extended to refer to real numbers) as well as allowing division and using function expressions. Thus, expressions such as \( 1/x, \sqrt{x^2 + y^2}, ye^x, \ln(1 + x), \sin(xyz) \), would be adjoined to the previous list.

It is also useful to allow expressions that involve no variables, that is, involving only constants and operations and functions applied to these. Thus, for example, the constants \( 0, 1, \sqrt{2} \) are expressions, as are \( 1 + 1 \), or \( e^2 \), or \( 2\pi + 3 \).
Each system has its own ingredients and rules of formation, but the general scheme of building complex expressions from simple pieces is common to all of these. All of this is defined precisely and in complete generality in a course in mathematical logic. We do not do this here, relying instead on the knowledge we have already gained from years of experience in working with mathematical expressions and with an informal understanding of the concept.

6. Predicates

A predicate $P$ in a mathematical system is a declarative assertion involving a finite number of expressions and relations in the system. If no logical operations appear in the predicate, i.e., it contains only expressions and relations, we may say that $P$ is a simple predicate or an atomic predicate. Just as in the case of statements in propositional calculus, more complex predicates are constructed by applying logical operations to simpler ones, starting with atomic predicates.

The variables in the expressions are drawn from a finite list, say $x_1, x_2, \ldots, x_n$, so we may write $P = P(x_1, x_2, \ldots, x_n)$ to emphasize the role of the variables. Not every expression contained in $P$ need make use of every variable. We assume that $x_1, x_2, \ldots, x_n$ range over various sets: $x_i$ ranges over the set $S_i$. It could be that all of the $S_i$ are equal to one given set $S$, or it could be that some or all of them are distinct from the others. This depends on the mathematical system we are considering and on the predicate $P$. We sometimes say that $P$ is a predicate over $S_1, S_2, \ldots, S_n$.

For example, in the theory of real numbers, we have simple predicates such as $e^x > 0$, $x^2 + y^2 = 1$, $\sin(x + y) = \sin x \cos y + \cos x \sin y$, and so on. In number theory, we have predicates such as $m | n$, $\ell + m \equiv n \pmod{17}$, $m > n$, and so on. In the first example, $x$ and $y$ are real variables, $e^x, \sin x, \cos y$, etc. are well-known functions, and the relations are $>$ and $\equiv$. In the second example, $\ell, m, n$ are variables and $|, \equiv (\pmod{17})$, and $>$ are relations.

It is assumed that a predicate $P$ is a meaningful assertion when each variable $x_i$ is interpreted as a general element of set $S_i$, $i = 1, 2, \ldots$, etc. For example, $x^2 + 2x - 3 = 0$ defines a predicate $P(x)$ over the real numbers. Note that it asserts something: namely, that the square of a real number plus twice that number minus 3 equals 0. However, it is not a statement, because its truth or falsity depends on the particular value of $x$.

The variables appearing in a predicate are of two kinds: those that have been quantified and those that have not. We describe quantification later. For now, we note only that quantified variables are often called bound variables, whereas unquantified ones are called free variables.

7. Logical operations and predicates

The logical operations of propositional calculus—negation, conjunction, disjunction, implication, identity, together with their iterations—can be applied to predicates just as they are to statements. Thus, for example, let $P = P(x_1, x_2, \ldots, x_k)$ and $Q = Q(y_1, y_2, \ldots, y_\ell)$ be predicates. We allow the possibility that some of the $y_j$’s equal some of the $x_i$’s. Then we can form new predicates

$$\neg P, \ P \land Q, \ P \lor Q, \ P \Rightarrow Q, \ P$$

involving the variables

$$\{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_\ell\}.$$

As stated in §6, these predicates are to be understood as declarative assertions about general elements of the sets over which the variables range. Note that these logical operations do not reduce the number of variables.

8. Specialization

Because predicates generally have variables, they allow logical operations that are not available to us in the case of statements. (More precisely, they are trivial in the case of statements.) The simplest such operation is that of specialization of a variable.

To describe this precisely, we begin with the idea of specializing a variable in an expression. Suppose that $E$ is an expression in which the variable $x$ appears, $x$ ranging over the set $S$. We may then write $E(x)$ to emphasize the appearance of $x$ in $E$ (although other unexhibited variables may also appear in $E$). We shall say that $x$ is specialized to a value $a$ in $E$ if every occurrence of $x$ in $E$ is replaced by the same symbol $a$, which represents a particular element $S$. (Note that it is important that this symbol not also be used in a different role in the expression!) A slightly different but similar locution is sometimes used for the same thing; namely, we may say that $E$ is specialized at $x = a$.

If $x$ is specialized to $a$ in $E$, we may then denote the specialized expression by $E(a)$. If $E$ is, indeed, an expression involving $x$ and other variables $y, z, \ldots$, then $E(a)$ is simply an expression involving $y, z, \ldots$, but no longer involving $x$. If $E$ involves only $x$ at the outset, then $E(a)$ is an expression involving no variables, i.e., it represents a specific element of $S$ or of some other set considered in the system.

For example, suppose $S$ is the set $\mathbb{Z}$ of integers and $E = E(x)$ is the expression $y - x^2 - 2x$. Then $E(1) = y - 3$. For another example, suppose $S$ is the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, $T$ is the set of positive reals, and $E(x) = e^{-x}$, where $x$ ranges over $\mathbb{N}$. Then, $E(3) = e^{-3} = (1/e)^3 \in T$. And so on.

Often this process is described as evaluating $E(x)$ at $x = a$.

Of course, we may specialize more than one variable at a time, as long as we take care to substitute each occurrence of the variables by the specific elements to which they are being specialized. The elements may or may not be distinct from each other, as the case may dictate.

For example, when $E = y - x^2 - 2x$, as above, we may write $E$ as $E(x, y)$ and then specialize to $E(-2, 0) = 0$ or to $E(1, 1) = -2$, etc.

If $E$ involves exactly $n$ variables, say $x_1, x_2, \ldots, x_n$, then it may be convenient to exhibit all of these in our notation, and we write $E = E(x_1, x_2, \ldots, x_n)$.

We now consider a predicate $P$ in which a free variable $x$ appears. That is, $x$ appears as an unquantified variable in one or more expressions that appear in $P$. We may write $P(x)$ to focus attention on $x$. Let $a$ be a particular element of the set $S$ over which $x$ ranges. We say that $x$ is specialized to $a$ in $P$ if $x$ is specialized to $a$ in every expression in $P$ that contains $x$. (Sometimes, instead, we may say that $P$ is specialized at $x = a$.) We may write the result as $P(a)$.

This is now a predicate in which $x$ no longer appears: it is a predicate in one fewer free variable, and we often say that the variable has been eliminated. If all the free variables in a predicate get eliminated, then, as we shall see, the result is a statement, with a definite truth-value.

This process may clearly be repeated any number of times, as long as there are free variables left. For example, consider the simple expression $E(x, y) : y = x^2 - 2x$ mentioned
above, and form the predicate $P(x, y) : y - x^2 - 2x = 3$. Then, $P(5, y)$ is the predicate $y - 5^2 - 2 \cdot 5 = 3$, and $P(5, 17)$ is the (false) statement $17 - 5^2 - 2 \cdot 5 = 3$.

Much the same thing will be seen in the case of quantification.

9. Logical Equivalence of Predicates

Just as with statements in Propositional Calculus, we are interested in when two predicates $P$ and $Q$ can be reasonably said to be logically equivalent. This motivates the following considerations.

Two preconditions clearly make sense. First of all, the predicates should both be part of the same mathematical context or theory (e.g., calculus, number theory, linear algebra, etc.). Secondly, both should involve the same free variables ranging over the same sets. (If the symbols used for the free variables in one predicate are different from those in the other, we assume that the variables in one of them can be re-labeled so that they are the same as corresponding variables in the other.)

With those preconditions satisfied, we can now define what it means for $P$ and $Q$ to be logically equivalent. For notational simplicity, we assume that the only free variables involved are $x$ and $y$, so that $P$ and $Q$ may be written as $P(x, y)$ and $Q(x, y)$, respectively.

Then, $P(x, y)$ and $Q(x, y)$ are logically equivalent—written $P \stackrel{\text{LE}}{\leftrightarrow} Q$—if and only if, for all possible specializations $x = a$ and $y = b$, the truth value of the statement $P(a, b)$ equals the truth value of $Q(a, b)$. Recall, in the chapter on the propositional calculus, we have seen that $P(a, b)$ and $Q(a, b)$ have the same truth value if and only if $P(a, b) \iff Q(a, b)$ is true, which we write more briefly as $P(a, b) \iff Q(a, b)$. So we may abbreviate the definition of logical equivalence of $P$ and $Q$ as follows: $P \stackrel{\text{LE}}{\leftrightarrow} Q$ if and only if $P(a, b) \iff Q(a, b)$, for all specializations $x = a$ and $y = b$.

Here are two simple examples. In both cases the variables are assumed to range over the real numbers. (a) Let $P(x, y)$ be the predicate $sin(x) = sin(y)$, and let $Q(x)$ be the predicate $x = y$. Clearly, the statements $P(a, a)$ and $Q(a, a)$ are both true, for every real number $a$. However, $P(0, \pi)$ is true, whereas $Q(0, \pi)$ is false. Therefore, it is not the case that $P \stackrel{\text{LE}}{\leftrightarrow} Q$. (b) Let $P(x)$ be the predicate $x^2 - 1 = 0$, and let $Q(x)$ be the predicate $(x = 1) \lor (x = -1)$. It is easy to see that both $P(x)$ and $Q(x)$ are true when $x$ equals 1 or –1, and both are false otherwise. So $P \not\stackrel{\text{LE}}{\leftrightarrow} Q$.

Thus the notion of logical equivalence of predicates is based on that used for statements. And, therefore, not surprisingly, it has most of the same properties of the earlier notion. For one example, a general substitution law holds, as the following exercise illustrates.

Exercise 3. Suppose that $K$ and $L$ are logical operations of the propositional calculus, each operating on the two atomic statements $P$ and $Q$. Suppose that the logical expressions $K(P, Q)$ and $L(P, Q)$ are logically equivalent, as defined in §7 of the Symbolic Logic I notes. Let $A$ and $B$ be any predicates, so that $K(A, B)$ and $L(A, B)$ are also predicates. Prove that $K(A, B) \stackrel{\text{LE}}{\leftrightarrow} L(A, B))$. (Hint: You may assume, for simplicity, that both $A$ and $B$ involve exactly the variables $x$ and $y$. You must show, for any specializations $x = a$ and $y = b$, that $K(A(a, b), B(a, b))$ has the same truth value as $L(A(a, b), B(a, b))$).

The following exercise shows that the relation of truth-equivalence of predicates depend on the choice of sets over which the variables range.
Exercise 4. Consider the following predicates:
\[ P(x, y) : y = 5 - x^2 \quad \text{and} \quad y > 0. \]
\[ Q(x, y) : y = 7 - 3x \quad \text{and} \quad 0 < x \leq 2, \]
where the variables \( x \) and \( y \) are assumed to vary over the positive integers.

(a) Verify that \( P \iff Q \).

(b) Suppose that, instead of assuming that the variables range over the positive integers, assume that the variables \( x, y \) range over the positive real numbers. Verify that \( P \iff Q \) is false.

(Hint: For (a), you have to show that the set of all \((a, b)\) that satisfy \( P(x, y) \) is the same as the set of all \((a, b)\) that satisfy \( Q(x, y) \) (assuming that \( a \) and \( b \) are positive integers). For (b), you have to show that these two sets are different (assuming that \( a \) and \( b \) are positive real numbers).)

10. Quantification

Quantification is a logical operation applied to predicates \( P \). The general scheme of things goes like this. A variable \( x \) is selected, and we quantify \( P \) with respect to \( x \). If \( x \) does not appear in \( P \), then the quantification operation is trivial: we get \( P \) again. If \( x \) does appear in \( P \), then we get a new predicate in which \( x \) has been eliminated. We can now, if we choose, quantify this new predicate again with respect to some other variable. And so on.

In contrast to the operation of specialization, in which several variables may be specialized at once and the order of specialization is irrelevant, quantification is applied to one variable at a time and the order of quantification, in general, does make a difference—as we shall see in the next section.

In what follows, we must distinguish two kinds of variables in \( P \): namely, those variables in \( P \) that we have already quantified earlier and those we have not. The former are called bound variables, the latter free variables. Quantification may be applied only to free variables, hence, no variable gets quantified more than once.

We now become more precise and describe the two kinds of quantification: universal and existential.

To fix notation, we suppose that \( P \) is a predicate containing a free variable \( x \) ranging over a set \( A \). It is possible that \( P \) has other variables as well, some bound, some free, but to signify our present interest in \( x \), we shall write \( P \) as \( P(x) \). The so-called scope of \( x \) consists of all occurrences of \( x \) in \( P \).

To quantify \( P(x) \) with respect to \( x \), we consider all possible specializations \( x = a \) in \( P \), each one giving a predicate \( P(a) \) in the remaining variables. Quantification with respect to \( x \) is an assertion about these \( P(a) \). There are two cases.

10.1. Universal quantification. For universal quantification, we form \((\forall x)P(x)\) (to be read “For all \( x \), \( P(x) \)”). This asserts all the specialized \( P(a) \).

If \( P \) contains free variables other than \( x \), say \( y, z, \ldots \), then \((\forall x)P(x)\) is a predicate in these remaining free variables, and again we say that the variable \( x \) has been eliminated. If \( x \) is the only free variable in \( P \), then \((\forall x)P(x)\) has no free variables, so it is a statement. Indeed, it is precisely the statement that asserts all of the statements \( P(a) \) simultaneously.

Here are two examples:
• Let \( P \) be the predicate \( x^2 + 1 \neq 0 \), and suppose that \( x \) ranges over the set of real numbers. Then, \( (\forall x)P(x) \) asserts: for every real number \( a \), \( a^2 + 1 \neq 0 \). This is certainly a true statement.

• Let \( P \) be the predicate \( 3x + 4y = 5 \), so that \( P \) has two variables \( x \) and \( y \). We suppose that they both range over the set of rational numbers. Then, \( (\forall x)P(x) \) is the assertion: for each rational number \( a \), \( 3a + 4y = 5 \). This is a predicate involving one variable \( y \). If \( y \) is subsequently specialized to some rational value, say \( b \), then the predicate becomes a statement asserting that, for each rational number \( a \), \( 3a + 4b = 5 \). Clearly, such an assertion is false no matter what specific rational number \( b \) represents.

Therefore, to repeat: With universal quantification we are asserting all of the specialized assertions \( P(a) \) at once. When \( x \) is the only free variable in \( P \), we see that \( (\forall x)P(x) \) is a true statement precisely when all of the \( P(a) \) are true simultaneously.

10.2. Existential quantification. For existential quantification, we form \( (\exists x)P(x) \) (to be read, “There exists an \( x \) such that \( P(x) \)”). This is the predicate that asserts at least one of the predicates \( P(a) \). (Note that it does not specify which ones.)

Again, if \( P \) contains other free variables \( y, z, \ldots \), then \( (\exists x)P(x) \) is a predicate in these remaining free variables. If not, then \( (\exists x)P(x) \) is the statement that “At least one of the statements \( P(a) \) is true.”

Here are two examples of existential quantification:

• Let \( P \) be the predicate \( x^2 + x + 1 = 0 \), where \( x \) ranges over the real numbers. Then \( (\exists x)P(x) \) asserts that the polynomial has a real root, which the student may easily check to be a false statement.

• Let \( P \) be the predicate \( x^2 + y^2 + 2xy - y + 3 = 0 \), where both \( x \) and \( y \) range over the reals. Then \( (\exists x)P(x) \) is a predicate in the variable \( y \). We may also write this as \( (\exists x)P(x,y) \) to signal the role that \( y \) plays. In any case, let us write this new predicate as \( Q(y) \). For any specialization \( y = b \), \( Q(y) \) clearly becomes \( Q(b) \)—i.e., \( (\exists x)P(x,b) \)—which is the statement that the polynomial \( x^2 + b^2 + 2xb - b + 3 = 0 \) has a real root. Using the quadratic formula, it is not hard to show that this statement is true when \( b \geq 3 \), and it is false when \( b < 3 \).

Therefore, to repeat: With existential quantification we are asserting at least one of the specialized assertions \( P(a) \) (but not specifying which one). When \( x \) is the only free variable in \( P \), we see that \( (\exists x)P(x) \) is a true statement precisely when at least one specialization \( P(a) \) is a true statement.

Exercise 5. Verify the last assertion in the second example above.

11. Changing the order of quantification

When we apply logical operations successively to predicates (or read a predicate that involves a succession of logical operations), we must be careful about the order in which the operations are applied. This is, for example, completely analogous to composing linear operators defined on a vector space, or to multiplying matrices. The order in which this is done will affect the answer.

In this section, we look at some examples of this in case the operations we exchange are both quantification operations. The following two sections show what happens when we try to exchange quantification with some of the elementary logical operations.
For our first example, let us deal with real variables again, and let $R$ be the predicate $x + y < 1$. Now consider the statement $(\forall x)(\exists y)R(x, y)$. To determine whether it is true or false, we begin at the left, just as with function composition. We must decide whether, for every real number $r$, the statement $(\exists y)R(r, y)$ is true. Now, in turn, this statement is true if we can find at least one real number $s$ such that $R(r, s)$ is true: that is $r + s < 1$. Clearly, this last is possible: just choose $s$ to be any real number $< 1 - r$. So, we have shown that $(\forall x)(\exists y)R(x, y)$ is true.

But notice what happens when the quantifiers are interchanged. We obtain the statement $(\exists y)(\forall x)R(x, y)$. Just by reading the statement —“There is a $y$ such that for all $x$, $R(x, y)$”— one can see that this is not the same as the previous statement. But, to clinch the matter, let us check its truth value. Again start on the left. To determine whether or not the statement is true, we must find at least one real number $u$ such that $(\forall x)R(x, u)$ is true. And to demonstrate this last, we must show that for any real number $v$, $v + u < 1$. But no matter what our choice of $u$, the real number $v = 2 - u$ violates the condition $v + u < 1$. Therefore, the statement $(\exists y)(\forall x)R(x, y)$ is false.

There are, however, certain cases of successive quantification in which the order does not matter. Here is a rule that covers these cases: Whenever two successive quantification are of the same type—i.e., both universal or both existential—then the order of quantification does not matter.

For example, consider the predicate $P(x, y) : x^2 + 2y - 6 < 0$, defined for $x$ and $y$ ranging over the real numbers. Then $(\exists x)(\exists y)P(x, y)$ is logically equivalent to $(\exists y)(\exists x)P(x, y)$. For each of these is true exactly when $P(a, b)$ is true for some real numbers $a$ and $b$ (in fact, precisely when the real numbers $a$ and $b$ satisfy $a^2 < 6 - 2b$) and false otherwise. For another example, let $Q(x, y)$ be the predicate $e^x + y = e^x \cdot e^y$, where $x$ and $y$ range over the reals. Then $(\forall x)(\forall y)Q(x, y)$ and $(\forall y)(\forall x)Q(x, y)$ both assert (truthfully in this case) that the statement $Q(a, b)$ is true for all real numbers $a$ and $b$.

12. The interaction between quantification and negation

Let $P$ be a predicate containing the free variable $x$ ranging over the set $S$ (and perhaps other variables). Then, the basic facts relating $\forall x$ to negation can be stated as two propositions:

1. Proposition. $\neg(\forall x)P(x) \iff (\exists x)(\neg P(x))$.

2. Proposition. $\neg(\exists x)P(x) \iff (\forall x)(\neg P(x))$.

Exercise 6. Prove each of the two propositions under the assumption that $x$ is the only free variable in $P$. (Hint: Note that in this special case, the two predicates exhibited in the first proposition are actually statements. So, it suffices to show that these two statements have the same truth value. This can be derived directly from the definition of quantification. Similarly for the second proposition.)

Notice that in both propositions, on the left-hand side, we are negating first and then quantifying, whereas on the right-hand side we are quantifying first and then negating. Therefore, to make a valid change of order in this case, we have to change the quantifier: in the first proposition, we change the universal quantifier to an existential one; in the second proposition, we do the reverse.

Here are some special cases of the above propositions. In the first four, De Morgan’s laws are also used. In the last two, one also makes use of the logical equivalence $\neg (A \Rightarrow B) \iff A \land \neg B$. 

A word of advice (which applies to many rules and formulas in mathematics): It is best not to try to memorize the following list. Rather, remember the principles by which the list is constructed. In this case there are two principles: (i) Exchanging the order of quantification and negation changes a universal quantifier to an existential quantifier and vice versa. (ii) Exchanging negation with an elementary logical operation follows the rules in the propositional calculus. Specifically, De Morgan’s Laws show that a disjunction is changed to a conjunction and vice versa.

- \( \neg(\forall x)(A(x) \land B(x)) \iff (\exists x)(\neg(A(x) \land B(x))) \)
- \( \neg(\forall x)(A(x) \lor B(x)) \iff (\exists x)(\neg(A(x) \lor B(x))) \)
- \( \neg(\exists x)(A(x) \land B(x)) \iff (\forall x)(\neg(A(x) \land B(x))) \)
- \( \neg(\exists x)(A(x) \lor B(x)) \iff (\forall x)(\neg(A(x) \lor B(x))) \)
- \( \neg(\forall x)(A(x) \Rightarrow B(x)) \iff (\exists x)(A(x) \land \neg B(x)) \)
- \( \neg(\exists x)(A(x) \Rightarrow B(x)) \iff (\forall x)(A(x) \land \neg B(x)) \)

To be more concrete, let us take the third of these, letting \( A(x) \) be the predicate \( x^2 - 3x + 2 = 0 \) and \( B(x) \) the predicate \( x^2 - 6x + 9 = 0 \). Then \( \neg(\exists x)(A(x) \land B(x)) \) asserts that it is not the case that the two polynomial equations have a root in common, whereas \( (\forall x)(\neg(A(x) \lor B(x))) \) asserts that, for every real number \( x \), either \( x^2 - 3x + 2 \neq 0 \) or \( x^2 - 6x + 9 \neq 0 \). Clearly these two statements assert the same thing, so they have the same truth-value.

13. The relationship between quantification and conjunction and disjunction

We begin with two cautionary examples.

First, suppose that \( A = A(x) \) is the predicate \( x^2 + 3x + 2 = 0 \) and \( B = B(x) \) is the predicate \( 4x - 7 = 0 \), where the variable \( x \) ranges over, say, the complex numbers \( \mathbb{C} \). Now consider the following statements:

1. \( (\exists x)(A(x) \land B(x)) \)
2. \( (\exists x)A(x) \land (\exists x)B(x) \)

Statement (1) asserts the existence of a real number which is a simultaneous root of the two equations, whereas statement (2) asserts only that each equation separately has a root. Clearly, these are two very different assertions, so it is not surprising that they have different truth values, statement (1) being false and statement (2) true.

Next, we’ll use the predicates \( C(x) : x < 3 \) and \( D(x) : x > 0 \), where \( x \) is a variable ranging over the reals, and we consider

3. \( (\forall x)(C(x) \lor D(x)) \)
4. \( (\forall x)C(x) \lor (\forall x)D(x) \)

In statement (3), we are first quantifying and then performing a disjunction; in (4), we are reversing the order. Statement (3) asserts that every real number is either less than 3 or greater than 0, which is certainly true. Statement (4) asserts that either every real number is \(< 3\) or that every real number is \(> 0\). Neither of these is true, so certainly statement (4) is false.

If we look closely at what “went wrong” in these examples, we can see that the problem involves the scope of the quantification operations. In statement (1), there is one instance
of quantification over the variable x, the scope of which involves both predicates A and B. In statement (2), there are two quantification operations, each with more limited scope (the one involving only A, the other only B). A similar observation applies to statements (3) and (4).

These examples show that we have to be careful about the scope of the quantifiers before trying to exchange them with conjunction or disjunction.

The observation about scope, however, shows us how we may do this in cases like those of statements (2) and (4). Let $Qx$ denote either $\forall x$ or $\exists x$, and let $E(x)$ and $F(x)$ be two predicates involving the free variable $x$ (and perhaps some other variables). We'll make use of the fact that we may change the name of the variable $x$ without affecting truth-values. We can choose any symbol instead of $x$ provided it does not already appear in the predicate.

So, suppose $y$ does not appear in $F$, and then replace $x$ by $y$. Then, $(Qx)F(x)$ asserts the same thing as $(Qy)F(y)$. Now make sure that $y$ is chosen not only to be different from any variable appearing in $F$ originally, but also different from any variable appearing in $E$. Then, the two predicates

\begin{align*}
(5) & \quad (Qx)E(x) \land (Qx)F(x), \text{ and} \\
(6) & \quad (Qx)E(x) \lor (Qx)F(x)
\end{align*}

can be replaced by

\begin{align*}
(7) & \quad (Qx)E(x) \land (Qy)F(y), \text{ and} \\
(8) & \quad (Qx)E(x) \lor (Qy)F(y),
\end{align*}

respectively, without changing their meaning.

Having done this, we can now see that, for each predicate, since the two quantifications involve different variables, we may move the second quantifier to the left, without affecting the meaning or truth value):

\begin{align*}
(9) & \quad (Qx)(Qy)(E(x) \land F(y)), \text{ and} \\
(10) & \quad (Qx)(Qy)(E(x) \lor F(y)).
\end{align*}

Therefore, a quantifier may be “moved to the left”, provided the quantified variable is given a new name different from those of other variables already appearing.

Note that if we apply this to statements (2) and (4), we do get all the quantifiers on the left, but there are still two quantifications required rather than just one as in statements (1) and (3). Statements (1) and (3) (which already have their single quantifier on the left) are not affected by this procedure.

The above procedure still works even when the second quantifier is not the same as the first. For example, we can replace $(\exists x)E(x) \land (\forall y)F(y)$ by $(\exists x)(\forall y)(E(x) \land F(y))$, etc.

Finally, the logical equivalence (f) in Exercise 11 of the propositional calculus notes

$$(P \Rightarrow Q) \iff (\neg P \lor Q),$$

together with the fact that substituting a predicate by a logically equivalent one does not affect truth values, shows that the relationship between quantification and implication may be understood in terms of relationship between quantification, negation, and disjunction. We don’t go any further into this, but we present the following exercise as an illustration.
Exercise 7. Let \( P(x, y) \) be the predicate \( x < y^{-1} \) and \( Q(x, z) \) the predicate \(-x = z^2\). Here we assume that \( x \) and \( z \) range over the real numbers and \( y \) ranges over the positive real numbers. Show that
\[
\left( \forall y \right) P(x, y) \Rightarrow \left( \exists z \right) Q(x, z) \iff \left( \exists y \right) \left( \exists z \right) \left( P(x, y) \Rightarrow Q(x, z) \right).
\]
(Hint: There are two ways to approach this problem. The first way is to use the discussion above to change the implication in the left-hand predicate to an expression involving disjunction, then to move the quantifier \( \exists z \) to the left in line what we discuss above, and finally, to change the resulting expression back to one involving implication. The other way is to use the definition of equivalence of predicates to show directly that both sides are equivalent — note that both sides are predicates in the free variable \( x \).)

Exercise 8. Formulate the following statements using predicates and quantifiers:

(a) If \( a, b, \) and \( c \) denote the lengths of the sides of a planar triangle, then \( a + b > c \).
(b) If \( a, b, \) and \( c \) denote the lengths of the sides of a planar right triangle, with \( c \) the largest, then \( a^2 + b^2 = c^2 \).
(c) If \( a, b, c \) and \( n \) are any positive integers and \( n > 2 \), then \( a^n + b^n \neq c^n \).

Now reformulate each of the above, if necessary, so that only negation, disjunction, and quantification are used.

Statement (a) is a special case of the well-known Triangle Inequality that students often see in a linear algebra course. Statement (b) is, of course, the Pythagorean Theorem, which is one of the cornerstones of geometry. And statement (c) is known as Fermat’s Last Theorem. It was proved in 1994 by Andrew Wiles, more than 350 years after it was conjectured by Pierre de Fermat.

Exercise 9. (a) Let \( f \) be a real-valued function defined on an open interval \( I \) of real numbers, and let \( a \) be an element of \( I \). The following statement expresses the fact that \( f \) is continuous at \( a \). Formulate that statement in terms of quantifiers and predicates: For every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that, for every real \( x \) satisfying \( 0 < |x - a| < \delta \), we have \( |f(x) - f(a)| < \epsilon \). Now formulate this so that only quantification, negation and disjunction are used and so that all quantification occurs on the left.

(b) Let \( f \) be a real-valued function defined on an open interval \( I \) of real numbers, and let \( a \) be an element of \( I \), as in the foregoing. Express the assertion that \( f \) is not continuous at \( a \) by applying negation to the last statement you obtained in the preceding exercise and exchanging negation with the other operations according to the rules we’ve derived. In the end, you should have a statement for which negation has been completely “distributed” (i.e., cannot be exchanged any further with other operations without reversing steps).

(c) Let \( f \) be a continuous real-valued function defined on a closed interval \( J \). Use quantifiers (on the left) and predicates to express the fact that \( f \) achieves a maximum value on \( J \).

(d) Suppose that \( g \) is a differentiable (hence continuous) real-valued function defined on an open interval \( I \). Use quantifiers and predicates to express the fact that if \( g \) achieves a maximum value at some element \( x \in I \), then the derivative \( f'(x) = 0 \).
14. Quantification and methods of proof

All of the methods of proof discussed in The Propositional Calculus carry over to the more general setting of the predicate calculus. Only a few additional points need to be noted now, in connection with quantification.

14.1. Universal quantification. Suppose we wish to prove a statement of the form $(\forall x)P(x)$, where the free variable $x$ ranges over a set $S$. As discussed earlier, this means that we must prove the statements $P(s)$ for every choice of $s \in S$. When $S$ is a small set, this may sometimes be possible by direct enumeration and checking. But often $S$ is a large set, such as the set of natural numbers or the set of real numbers. In those cases a different approach is needed.

In the case of the natural numbers, a standard method is the method of mathematical induction. Students are already familiar with simple versions of this from, say, calculus courses, but in any case, we go into this method more fully in the chapter The Natural Numbers.

When induction does not seem appropriate or adequate, the usual method is to choose an arbitrary (i.e., general) element $s$ in $S$ and then prove $P(s)$. The choice of an arbitrary element $s$ means that the information about $s$ contained in the proof can only consist of properties that it has by virtue of its membership in $S$. Therefore, the proof applies simultaneously to every particular element in $S$.

Example: Let $x$ be a variable ranging over the real numbers. Let us prove that $(\forall x)((x > 1) \Rightarrow (x^3 > x^2))$.

We choose an arbitrary real number $s$. To prove that the implication $(s > 1) \Rightarrow (s^3 > s^2)$ is true, we need only show that it is not falsified. If $s \leq 1$, the implication is vacuously true, so we may then restrict attention to those cases in which $s > 1$. We must check that in those cases, $s^3 > s^2$. Since $s > 1$, we know that $s$ is positive. Using facts about real multiplication, we multiply both sides of the inequality $s > 1$ by $s$ and obtain $s^2 > s$. Now multiply this last inequality by $s$ to obtain $s^3 > s^2$. Therefore the implication is true.

The reader may have noticed that a slight shortcut is possible. Namely, since in the cases in which $s$ specializes to a number $\leq 1$, the implication is vacuously true, we don’t really need to bother checking anything. That is, we may as well assume the hypothesis $s > 1$ right away and proceed from there. This is a general feature of proofs of implications in which the hypothesis restricts the range of the variable.

The actual technique that you use to prove the universal statement will depend, of course, on the statement. Some suggestions about how to proceed appear in the chapter The Propositional Calculus.

Exercise 10. (a) Prove: For all positive, real numbers $a$ and $b$,

$$\frac{1}{2}(a + b) \geq \sqrt{ab}.$$ 

(b) Prove: For all $x \geq 1$, $x \geq 1 + \ln(x)$.

14.2. Existential quantification. To prove a statement of the form $(\exists x)P(x)$, where again $x$ ranges over $S$, we must show that, for some element $s \in S$, $P(s)$ is true. This may be accomplished by one of two methods.
In the first method, we produce such an element \( s \) explicitly, which we describe by saying that we find \( s \) or we construct \( s \). This itself is a two-step process: (1) The element \( s \) must be found or constructed, and then (2) the truth of \( P(s) \) must be demonstrated.

In the second method, the existence of \( s \) is proved, either directly as a consequence of some known result or indirectly. In the indirect proof, we assume that there is no such \( s \) and derive a contradiction. In symbolic terms, we prove that \((\forall x)\neg P(x) \Rightarrow C\), where \( C \) is a suitably formulated contradiction.

Here are two examples that illustrate these methods.

**Example:** Let \( P(x) \) be the predicate \( x^2 - 2x - 3 = 0 \), where \( x \) is a variable ranging over the real numbers. The statement \((\exists x)P(x)\) affirms that the equation has at least one real solution. It is this solution that we must find or construct. The student has already learned how to do this in a beginning algebra course. In general, one can try to factor the left hand side into linear factors. Or failing that, one can “complete-the-square” on the left-hand side. Or one can simply recall the quadratic formula and write down the roots. Even guessing is legitimate, though unless one has good reasons for a guess this method is not usually optimal. In the end, any of these methods should produce two possible values for a solution \( x = s \): namely, \( x = 3 \) and \( x = -1 \). This is the first step.

It remains to verify that the predicate is true for one or another of these values. This can be done by evaluating the equation at 3 or at \(-1\). Alternatively, often the method used to find the answer (e.g., factoring or completing the square) has steps that are reversible, so the verification can be done by just running the steps in reverse. This step is usually left out, since it is routine, but it is important to understand that, from a logical point of view, it needs to be there, even if just in background. So, after solving an equation as Step 1 in the proof of an existence statement, students in this course should either evaluate the equation at the found solution or say something like: *Since the foregoing steps are reversible, the value \( x = s \) does satisfy the equation.*

Although the foregoing procedure is the simplest for this particular problem, there is also a method that proves \((\exists x)P(x)\) directly, a method that can be applied to more complicated equations for which there are no nice algebraic methods or formulas. This method uses the Intermediate Value Theorem from calculus. We write \( f(x) = x^2 - 2x - 3 \) and observe that this defines a continuous function of a real variable. We then compute \( f(0) = -3 \) and \( f(5) = 12 \): note that \( f(0) \) is negative and \( f(5) \) is positive. The Intermediate Value Theorem then implies that there is a real number \( s \) between 0 and 5 for which \( f(s) = 0 \).

**Example:** This example also uses facts from calculus, including the Intermediate Value Theorem. The student is assumed to be familiar with these and should assume them while digesting the argument. Let \( I \) denote the set of all real numbers \( x \) satisfying \( 0 \leq x \leq 1 \). \( I \) is often called the unit interval. Suppose that \( f : I \to I \) is any given continuous function. We shall prove that \( f \) has a fixed point. That is, we prove the statement \((\exists x)(f(x) = x)\), where we assume that the variable \( x \) ranges over \( I \). In this case, we give an indirect proof. We assume that the statement is false and derive a contradiction.

By the earlier discussion, the negation of \((\exists x)(f(x) = x)\) is \((\forall x)(f(x) \neq x)\). Consider the function \( g \) given by the equation \( g(x) = f(x) - x \), for all \( x \in I \). Our assumption implies that \( g(x) \neq 0 \), for all \( x \), hence the same is true for the absolute value \( |g(x)| \). Finally, define the
function $h : I \to \mathbb{R}$ by the equation $h(x) = g(x)/|g(x)|$, for all $x \in I$. Since the denominator is never zero, basic facts about continuous functions proved in a calculus course imply that $h$ is a continuous function. Notice that, by definition, the value $h(x)$ is either equal to 1 (when $g(x) > 0$) or $-1$ (when $g(x) < 0$). $h$ never assumes a value other than these two. So far we have not reached a contradiction, because it is possible that $h$ is simply the function that is constantly equal to 1, or the function that is constantly equal to $-1$. However, this is not the case, as can be seen by evaluating $g(0)$ and $g(1)$. In fact, $g(0) = f(0) - 0 > 0$, because $f(0) \in I$ and $f(0) \neq 0$, by assumption. Further, $g(1) = f(1) - 1 < 0$, because $f(1) \in I$ and $f(1) \neq 1$, by assumption. It follows from what was said above that $h(0) = 1$ and $h(1) = -1$. The Intermediate Value Theorem then implies that there must be a real number $x \in I$ such that $h(x) = 0$. But we have seen that $h$ assumes only the values $\pm 1$. So, we have arrived at the promised contradiction.

This example is a special case of what is known as Brouwer’s Fixed Point Theorem, which belongs to a branch of mathematics known as topology. We have left out a number of smaller steps for the sake of brevity; this can make it tougher going for the reader. Most published mathematics, whether in class notes, textbooks, or published papers make such omissions. The higher the level of mathematics, the more omissions there will be. Therefore, it is very important at this stage for the student to get in the habit of reading proofs accompanied by writing material (e.g., pencil or pen and paper). Each time there are steps in the proof that merit checking or filling in, the student should do so.

Even though the proof above is more complex and requires more ingenuity than the usual proof we shall encounter, it is a very good example of an indirect proof that proves an existence statement.

**Exercise 11.** Let $p(x)$ be a polynomial of degree $n$, with top-degree coefficient 1. Define $q(x)$ for $x \neq 0$ by the equation $q(x) = p(x)/x^n$.

(a) Prove that $\lim_{x \to \infty} q(x) = 1$ and $\lim_{x \to -\infty} q(x) = 1$.

(b) Prove that $p(x)$ and $x^n$ have the same sign when $|x|$ is very large.

(c) Prove that $n$ odd $\Rightarrow (\exists x)(p(x) = 0)$.

**14.3. Concluding practical remarks.** The discussion above focused on only the simplest kinds of quantified statements that you may be called upon to prove. More complicated statements might have multiple quantifications or propositional connectives (i.e., conjunction, negation, etc.) intertwined with various predicates. These can always be untangled and reduced to a step-by-step consideration of the simpler forms we discuss above and in previous sections. Going into all these possibilities in detail would take us too far afield with not much practical benefit.

One practical problem that a student might confront is that statements that need to be proved are not always in nice explicitly quantified form even though there is implicit quantification occurring. For example, when you are asked to solve a certain equation, this is tantamount to proving an existential statement. Or, suppose you are given the following: “Let $p$ be an odd prime. Prove that $2^p - 2$ is divisible by $p$.” You are being asked to prove
a universal statement (i.e., for all odd primes \( p, \ldots \) etc.) This is largely a matter of getting used to the style of mathematical writing. It is worth paying close attention to this, since existential statements require different methods of proof from those required by universal statements.

A remark related to the foregoing caveat occurs when you are asked to prove or disprove a certain general statement. We have already given a suggestion as to how to proceed in *The Propositional Calculus*, §9 (c), “What to do when you are stuck,” item (iv). Here, we elaborate slightly.

As an example, you may be given what looks like it might be a trig. identity and you are asked to prove or disprove it. What do you do?

First recognize that the statement, say a trig identity or trig. identity look-alike, is a universal statement: e.g., For all real \( x \) and \( y \), \( \cos(x + y) = \cos(x) \cos(y) - \tan(x) \tan(y) \). Next, recognize that to prove this universal result, you need to work with general real numbers \( x \) and \( y \), not with specifically chosen ones. Third, recognize that when you disprove this universal result, you are proving the negation of a universally quantified statement. Therefore, you can exchange the order of quantification and negation, but you need to change the universal quantifier to an existential one (cf. §12). This means that, to disprove the universal assertion you need to find or construct specific values of the variables for which the identity fails.

Inspect the given statement to see if you can find any clues as to whether it is true or false. Maybe it’s similar to something you’ve seen before. Maybe you have a hunch. That’s good. Go with your hunches...at least for a certain amount of time. But, what do you do if you have no hunch? Or your earlier hunch failed? Well, it is usually much easier to calculate some specific values than it is to prove a general assertion. So, if you don’t know what to do, try to “plug in” some values and see what happens. There are two possibilities: (1) After a small number of trials, the equation fails. Good. You’ve proved the negation. You’re done. (2) After a small number of trials, the equation still holds. Okay. Not so bad. Don’t panic. But don’t stop here. You haven’t proved anything. At least not what you need to prove. But, you have gathered some evidence that the equation might be universally true (particularly if you chose your values relative randomly). Therefore, you can proceed to attempt a proof of the identity with a certain amount of confidence that this is the direction to go. (Of course, none of this advice tells you anything about trigonometry. That will be your problem.:-) )

Just for fun, consider the possible trig. identity above, taking the position that we don’t know whether it’s true or false. Let’s choose some values for \( x \) and \( y \). We need to pick values for which we can compute the expression, but the values shouldn’t be too trivial (e.g., like 0), since then we might get an equality for trivial reasons...a false positive, so to speak. So, let’s take \( x = \pi/4 \) and \( y = \pi/4 \). Then, \( \cos(\pi/4) = 1/\sqrt{2} \), \( \tan(\pi/4) = 1 \), \( \cos(\pi/4 + \pi/4) = \cos(\pi/2) = 0 \). Plugging in these values, we get \( 0 = (1/\sqrt{2})^2 - 1 \cdot 1 = -1/2 \),
which is clearly false. This proves that the equation is not an identity. (By the way, if we had picked $x = 0 = y$, we would have obtained $1 = 1 \cdot 1 + 0 \cdot 0$, which is true but misleading.)

This concludes our brief foray into the foundations of the predicate calculus. For more advanced work in this area, consider taking Math 481 or Math 483. For further practical tips in dealing with quantifiers in proofs, see Solow’s book, Chapters 4 through 7.