Glossary of Logical Terms†

The following glossary briefly describes some of the major technical logical terms used in this course. The glossary should be read through at the beginning and can then be consulted again as needed. The organization is logical rather than alphabetical.

1) In the beginning...

Definitions: In mathematics, we need to be as precise as possible with the terms that we use, since even one instance of imprecision can un-couple an entire chain of reasoning. So mathematicians make an effort to be very explicit and careful about the definitions of terms, particularly when working with something new, i.e., beyond the underlying context. A definition is an introduction and explanation of terminology or notation used in the subsequent mathematical text. Sometimes it can repeat a common, widely held understanding, similar to what we find in a dictionary definition. But, at other times, it is an explicit device used by the author/mathematician/student as a shorthand tool to refer to a particular new idea that they are working with. A definition is a matter of convention, to some extent arbitrary, expressing a definite understanding between author and reader which allows the mathematical conversation to proceed.

In presenting the terms that follow, we are giving definitions in the sense just described.

True and false statements: Mathematicians are constantly dealing with assertions (meaningful, declarative sentences) that correctly or incorrectly describe some underlying mathematical reality or context. Such an assertion is called a true statement when it does give a correct description and a false statement when it does not. In general, a meaningful, declarative sentence that is either true or false is called
simply a **statement**. (Some mathematicians and logicians use the term “proposition” for this, but we will not.)

The two important aspects of this notion of a mathematical statement are: (1) it is either true or false but not both, and (2) its truth (or its falsity) is a feature of the statement itself and does not depend on our ability to verify it.

The word “true” (and the word “false”) may also be used in a provisional sense, as when we assume that a certain statement is true and then reason based on this assumption. Paradoxically, such usage is valid even if our original assumption is counterfactual. For example, we may assume that a statement is true and then use reasoning to show that this assumption was incorrect. Indeed, this constitutes a useful technique of proof—proof by contradiction—that we’ll be discussing further later.

**Axiom:** A foundational statement or basic stipulation or assumption, taken to be true and used as a basis for further reasoning.

**Postulate:** A stipulation or assumption similar to an axiom but with a slightly more provisional status.

### 2) Getting going...

Statements, axioms, etc., are static entities, but mathematics is dynamic. New mathematical statements are constantly being produced. It is, therefore, very important that there be clearly defined rules for deriving new statements from old.

**Rule of inference:** A rule that allows us to proceed from a given statement to a subsequent statement in such a way that whenever the given statement is true, then the subsequent statement is also true. (See **modus ponens** below.)

**A caveat, however:** if the given statement is false, then there is no predicting the truth or falsity of the subsequent statement.

A sequence of applications of rules of inference is sometimes called a **valid chain of reasoning**.

**Proof:** A proof is a method for establishing the truth of a mathematical statement by arriving at it via a valid chain of reasoning from a set of
mathematical statements that are known to be true. One of the goals of this course is to learn various techniques for producing such valid chains of reasoning. A statement is called **provable** (relative to a given set of statements that are known or assumed to be true) if it is the end result of a proof (that starts with the given set). Thus, every provable statement is true (provided the statements in the given set are true). Note, however, that we are not asserting that every true statement is provable.

3) Where are we headed?

The goal of mathematics is to obtain a certain kind of specialized knowledge, much of which is formulated in terms of true mathematical statements. Some of these statements can be very elementary or obvious and others can be highly complex or surprising. Certain terminology has been adopted to signify the relative importance of such statements.

**Theorem:** A theorem is a true, *significant* mathematical statement whose truth has been established by a proof. People may differ as to the significance of this or that true mathematical statement, but most mathematicians agree that, to qualify as a theorem, such a statement must have import or application beyond the immediate context in which it appears.

**Proposition:** This term is often used for a “lesser” theorem, that is, for a true mathematical statement of middle-level significance. The term is also sometimes used instead of the word “statement,” as already mentioned, but we won’t use it in this way in this course.

**Lemma:** This term refers to a true mathematical statement, established via a proof, whose main importance is that it forms a steppingstone to a proposition or a theorem. A lemma does not usually have significance beyond this and is often of an elementary character.

**Corollary:** This is usually a theorem that is an immediate consequence of another theorem. So, we say “Statement B is a corollary of Theorem A.”

**Conjecture:** This term frequently appears in mathematical literature to designate a statement whose truth appears to the author to be very likely but which has not yet been established (by a proof).
Exercise 1. Which of the following are statements and which are not? Explain your choices.

(1) All men are mortal.
(2) $1 + 1 = 2$.
(3) $1 + 1 = 3$.
(4) Sam ate a sandwich.
(5) What time is it?
(6) Hands up!
(7) This sentence is false.
(8) The dog cat pizza votes swimming.

Exercise 2. (A discussion problem.) Mathematics also proceeds by paradigms that do not explicitly involve axioms, inferences, proofs, and theorems. For example, much of mathematics is devoted to solving various kinds of equations. Consider, for example, the simple equation $2x + 3 = 7$. Can you show how solving this equation can be formulated as a theorem that can be proved?

4) Some of the pieces

Statements, whether they occur in mathematics or not, can often be broken into simpler subsidiary statements that are linked by certain logical connectives. The original statement is then sometimes called a compound statement. The following list describes these briefly; more details will be presented later.

Throughout the following, we let $P$ and $Q$ stand for any given statements.

**Negation:** *It is false that* $P$ or, more briefly, *not* $P$. This asserts the contrary of what $P$ asserts and is called the *negation* of (the subsidiary statement) $P$. (Negation is called a logical connective even though it applies to only one statement.)

For good English usage, we may have to modify the resulting statement. For example, if $P$ is the statement “I can’t hear you,” then “It is false that I can’t hear you” is grammatically correct but awkward, and “not I can’t hear you” is not grammatically correct. So we may modify these to something like “I can hear you.” Similar common-sense modifications may be used for the other connectives.
Conjunction: $P$ and $Q$. This asserts both $P$ and $Q$ and is called the conjunction of (the subsidiary statements) $P$ and $Q$.

For example, the conjunction of the two statements, “The moon was full last night” and “The Knicks beat the Bullets,” is the assertion, “The moon was full last night and the Knicks beat the Bullets.”

More generally, when a statement asserts every one of a set of subsidiary statements, we say that the statement is the conjunction of the subsidiary statements.

Disjunction: $P$ or $Q$. This asserts that at least one of the two (subsidary) statements $P$ or $Q$ is true and is called the disjunction of the statement $P$ and the statement $Q$.

Notice that this notion of disjunction includes the case in which both $P$ and $Q$ are true and is sometimes called inclusive disjunction. In another form of disjunction, called exclusive disjunction, exactly one of $P$ or $Q$ is asserted but not both. For the most part, we do not use this form of disjunction, but when we do, we’ll explicitly call it exclusive disjunction.

As an example of disjunction, we use the statements just discussed above: the statement “The moon was full last night or The Knicks beat the Bullets,” is the disjunction of the statement “The moon was full last night” and the statement “the Knicks beat the Bullets.”

More generally, when a statement asserts that at least one of a set of subsidiary statements is true, then we say that the statement is the disjunction of the subsidiary statements.

Implication: $P$ implies $Q$. This asserts that $Q$ is true whenever $P$ is true. However, it asserts nothing about the truth or falsity of $Q$ in the case that $P$ is false.

Implication can have a variety of different English forms. In addition to “$P$ implies $Q$,” it is often expressed as “if $P$ then $Q$.” We’ll mention other forms later.

Using the (admittedly silly) example given earlier, we can consider the implication “If the moon was full last night, then the Knicks beat the Bullets.”

For the next five items, we shall be referring to the implication $P$ implies $Q$.

Hypothesis: $P$ is called the hypothesis of the implication.
Conclusion: $Q$ is called the conclusion of the implication.
Converse: $Q$ implies $P$ is called the converse of the implication.
Inverse: not$P$ implies not$Q$ is called the inverse of the implication.
Contrapositive: not$Q$ implies not$P$ is called the contrapositive of the implication.

Tautology: Compound statements may be formed by repeatedly applying one or more of the foregoing connectives to various subsidiary statements. Sometimes, the very logical structure of the compound statement forces it to be true no matter what the truth status of the subsidiary statements. In this case, we call it a tautology. For example, the disjunction of the two statements “The moon was full last night” and “It is false that the moon was full last night” is a true statement whether or not the moon was full last night.

Contradiction: This is defined to be a compound statement which is the opposite of a tautology: it is always false no matter what the truth status of the subsidiary statements. Another way of saying this is that a contradiction is the negation of a tautology, and a tautology is a negation of a contradiction.

The simultaneous assertion of any statement and its negation, i.e., for any statement $P$, the statement $P$ and not$P$, is an example of a contradiction, as follows immediately from our concept of truth. $P$ and not$P$ is perhaps the form of contradiction most widely used. Indeed, the literal meaning of the verb “to contradict” is to assert the contrary of (what has been asserted), which is precisely what $P$ and not$P$ does.

5) More terminology from the propositional calculus...

Truth-value: A numerical value, usually 1 or 0 assigned to a statement to indicate whether it is true or false. The truth-value of a compound statement can be algebraically expressed in terms of the truth-values of its subsidiary statements provided one uses ‘mod two’ arithmetic.

Truth-table: Because the truth-value of a compound statement $S$ depends only on the truth-values of its subsidiary statements $S_1, S_2, \ldots, S_r$, we can construct a table that lists all $2^r$ possible truth-values for $S_1, S_2, \ldots$ and, next to each, lists the truth-value of $S$. This is called the truth-table of $S$. 
**Logical operation:** A procedure that starts with some given statements and applies a finite number of logical connectives to them to obtain a compound statement.

**Logical expression:** A logical operation may be displayed symbolically by using logical connectives and parentheses, subject to various formation rules. This display is a logical expression.

**Atomic statement, atom:** A statement represented as an ingredient in a logical expression that is not viewed as being reducible into further subsidiary statements.

**Logical equivalence:** Two logical expressions are said to be logically equivalent if they have the same atoms and if any assignment of truth-values to these atoms produces the same truth-value for each expression. This can be checked by comparing truth-tables.

**Modus ponens:** The most basic rule of inference: If statements $A$ and $A \Rightarrow B$ are both true, then we may infer that $B$ is true.

6) Terminology from the predicate calculus

By itself, the propositional calculus is not refined enough for the needs of mathematics, especially modern mathematics, in which the concepts of a set and a variable ranging over a set are central. The predicate calculus builds on the propositional calculus to accommodate these ideas. Here, we shall assume the basic terminology of set theory, focusing only on those terms needed to describe concepts in the predicate calculus.

**General element, arbitrary element:** Given a set $S$, we often want to focus on one of its elements without specifying which one. We use only the properties the element has by virtue of its membership in $S$, not any further identifying properties. In that case, we talk about a general element of $S$ or an arbitrary element of $S$. For example, we may say, “Consider an arbitrary real number, $x$.” Then it would not be correct to imagine that $x = 1$ or $x = \sqrt{2}$ or $x = \pi$. Although these are real numbers, we know more about each one of them than that.

**Particular element, specific element:** The real numbers $1, \sqrt{2}, \pi$ just listed are examples of particular real numbers or specific real numbers. That is, each of them is a unique element in $S$ that we are specifying.
**Variable:** A linguistic object, usually represented by some symbol, that refers to a general element of a set. Thus, when we say “Let \( x \) vary over \( S \),” we are using the symbol “\( x \)” to refer to a general element of the set \( S \). The linguistic object—the variable— in this case is \( x \).

**Constant:** Also a linguistic construct, used to refer to a particular element of a set. Sometimes, the particular symbol used for a constant spreads among practitioners to the point of becoming a (virtually) universal name for that object: e.g., “0,” “1,” “\( e \).”

**Expression:** A linear chain of symbols representing constants, variables, functions, operations, as well as blanks (spaces), parentheses, brackets, etc. When such a chain satisfies certain rules of formation, which are specific to the mathematical subject at hand, then it is said to be *well-formed*. For example, in ordinary arithmetic, \( x + \sqrt{y^2 - 13}b \) and \( 1 + 1 \) are well-formed expressions, whereas \( x + +13b \) is not.

**Predicate:** An assertion composed of well-formed expressions and relations, with the property that, for any assignment in the predicate of one constant to each variable (the same constant in all of the occurrences of the variable), one gets a statement (in the sense of the propositional calculus...i.e., it is either true or false). For example, \( y + 2x = e^x z \) is a predicate involving the variables \( x, y \) and \( z \), the constants 2 and \( e \), the operations + and exponentiation, and the relation =. When \( x = 2, y = 3, z = 4 \), we obtain the statement \( 3 + 2 \cdot 2 = e^{2\cdot 4} \), which is false. When \( x = 1, y = 2, z = \ln(4) \), we get a true statement, etc. For another example, this time not involving =, consider \( 2^{2^x} > x + 3 \) and \( 5 + 3 \leq 17 \).

By definition, a predicate with no variables is just a statement...so the notion of a predicate generalizes the notion of a statement.

All the logical operations introduced in the propositional calculus can be applied to predicates. However, predicates are more general than statements, and so they have a richer structure that allows some further operations. If applied to a statement, these operations reduce to the identity operation.

**Specialization:** The first of these is called *specialization*. In specialization, we select some variable, say \( x \), which ranges over a set \( S \), and we choose a particular element \( a \in S \). Given a predicate involving the variable \( x \), we then replace every occurrence of \( x \) in the predicate by
the constant $a$: we have *specialized* the predicate at $x = a$. The result will be another predicate in which the variable $x$ no longer appears...it has been eliminated. See *The Predicate Calculus* for further details.

**Universal quantification:** This is an operation that also applies to a variable, say $x$, occurring in a predicate, say $K$. Call the predicate $K(x)$ to indicate that $x$ occurs in $K$. When we apply the *universal quantifier* $\forall x$ to $K(x)$, we are considering the specialization of $K(x)$ at $x = a$, written $K(a)$, *for each* $a \in S$. Each of these $K(a)$’s is a predicate in which $x$ no longer appears. When we universally quantify, we are asserting all the $K(a)$’s simultaneously! For example, consider the predicate $x + 1 > x$, where $x$ ranges over the real numbers. For any real number $a$, we have the specialization $a + 1 > a$. The universal quantification of this predicate, $\forall x)(x + 1 > x)$, asserts *all* of the inequalities $a + 1 > a$ simultaneously. This is a meaningful assertion, which no longer involves any variables: so it is a statement. Clearly it is a true statement. Thus, we say that $\forall x)(x + 1 > x)$ is true.

**Existential quantification:** There is one more operation that we apply to predicates. Using the notation above, we wish to apply the *existential quantifier* $\exists x$ to $K(x)$. We again consider all the specializations $K(a)$, only now we do not wish to assert all of them simultaneously. Rather, we wish to make the assertion that *at least one of them holds*. That assertion is abbreviated by $\exists x)K(x)$ and is usually expressed as “There exists an $x$ such that $K(x)$.” For example, let $K(x)$ be the predicate $x - 2 > 0$, where we are dealing with the arithmetic of real numbers. Specializing at the particular real number $x = a$ yields the statement $a - 2 > 0$. The existential quantification of the predicate $K(x)$, $\exists x)K(x)$, is the assertion that at least one of the inequalities $a - 2 > 0$ is true. Choosing, say $a = 8$, we get the true inequality $8 - 2 > 0$, so $\exists x)K(x)$ is true. (Of course, by the properties of real numbers, the inequality $a - 2 > 0$ holds for infinitely many choices of $a$, which is more than we need in this case.