for credit you must show your work

1. a) (10 points) For the equation \( y' = 2 - y \) with \( y(0) = 3 \), use the Picard method of successive approximations,

\[
y_{n+1}(t) = y_0 + \int_0^t f(y_n(s)) \, ds,
\]

to find approximate solution functions \( y_1(t) \) and \( y_2(t) \).

b) (10 points) Also find the exact solution. [Since the answer might be on your formula sheet, you must either: derive it somehow, or: explicitly check that your solution formula has both of the required properties.]

2. a) (10 points) For the same initial value problem as in Problem 1, use the Euler numerical method,

\[
y_{n+1} - y_n = f(y_n),
\]

to find the numbers \( y_1 \) and \( y_2 \), using stepsize 0.1.

b) (10 points) Sketch a direction field for the equation which shows dots for the numbers you found in part (a), and also shows the equilibrium solution, and also shows at least 10 slope marks in various places with roughly the right slope.

3. a) (12 points) Solve \( y'' + 2y' + 10y = 0 \) with initial values \( y(0) = 0 \) and \( y'(0) = 7 \).

b) (4 points) Verify by substitution that the function you found in part (a) really is a solution to the differential equation.

c) (4 points) Print your name and section number on your exam booklet.

4. a) (12 points) Solve the equation \( y' = -3(y - 100e^{-t} - 200) \), with \( y(0) = 20 \), which we view as a type of Newton Law of Cooling for an object of temperature \( y(t) \), in an environment which is also changing.

b) (4 points) What is the temperature of the environment in part (a)?

c) (4 points) Use either your solution of part (a) or a direction field to answer: Will the object ever have temperature \( y = 20 \) again after the initial time?

5. [Parts (a), (b), and (c) of this problem are not related to each other.]

a) (12 points) In our description of escape velocity we assumed there is a trajectory for which the distance from Earth to rocket \( r(t) \to \infty \) and the velocity \( r'(t) \to 0 \) as \( t \to \infty \). Find a solution to the equation we derived, \( r' = \frac{\sqrt{2GM}}{\sqrt{r}} \), and explain why it has those two limits.

b) (4 points) Joe claims that, according to the Existence and Uniqueness Theorem, the graphs of any two solutions to \( y' = t^2 + y^2 \) cannot cross each other. If Joe is right, explain why, and if wrong, give an example of crossing solutions.

c) (4 points) Verify that \( T^7 V^3 = c \) is a solution to \( 7VdT + 3TdV = 0 \).
some short answers and hints

1a) \( y_2(t) = 3 - t + \frac{1}{2}t^2 \)
1b) \( 2 + e^{-t} \)
2a) \( y_2 = 2.81 \)
2b) The dots ought to lie above and decrease toward the equilibrium solution \( y = 2 \)
3a) \( \frac{7}{3}e^{-t}\sin(3t) \)
4a) \( 200 + 150e^{-t} - 330e^{-3t} \)
4b) For this you have to compare the equation with the standard Newton Law of Cooling to see what has been changed
4c) No, and probably the direction field is the best thing to look at. Be sure to sketch the environment temperature as a guide.

5a) The equation is separable, and I found \( \left(\frac{r_0^{2/3}}{3} + \frac{\sqrt{2GMt}}{3}\right)^{2/3} \)
5b) If solutions cross at some point, see whether the Theorem applies at that point.
5c) Since \( T^7V^3 = e \) is an equation, not a function, you have to know what the word “solution” means in this context. It suffices to check the differential of the function \( T^7V^3 \). But you could instead solve for \( T \) in terms of \( V \) or vice-versa and work it that way.