Your name: ____________________________________________

Math 2930, Final Exam
Monday Aug 6th, 2012. 8:30–11:00 AM

This exam should have 7 pages, with 7 problems adding up to 150 points and one extra credit problem worth 15 points. *On that extra credit “bonus” problem*, only if a substantial portion of your response is correct will you receive any credit for your work. It is therefore recommended that the problem be attempted last, and only if you have time remaining.

The last page is blank and can be used as scrap paper for computations and checking answers.

No calculators or books allowed.

To improve your chances of getting full credit (or maximum partial credit) and to ease the work of the graders, please:

- write clearly and legibly;
- box in your answers;
- simplify your answers as much as possible;
- explain your answers as completely as time and space allow.
- any formula may be used, but *must* be clearly stated before use.

| Problem 1 |  /20 |
| Problem 2 |  /20 |
| Problem 3 |  /15 |
| Problem 4 |  /15 |
| Problem 5 |  /25 |
| Problem 6 |  /30 |
| Problem 7 |  /25 |
| BONUS:    |  /15 |

TOTAL: /150

Academic Integrity is expected of all students of Cornell University at all times, whether in the presence or absence of members of the faculty.

Understanding this, I declare I shall not give, use, or receive unauthorized aid in this examination.

______________________________
Signature of the Student
1. A tank contains 500 L of pure water. Brine that contains 0.05 kg of salt per liter flows into the tank at the rate of 8 L/min. The solution is kept thoroughly mixed and drains from the tank at the rate of 8 L/min. Let \( y(t) \) be the amount of salt in the solution at time \( t \), measured in kg; time is measured in minutes. What differential equation is satisfied by \( y \)? Without solving this equation, guess the value of \( \lim_{t \to \infty} y(t) \).
2. Find all solutions to the ODE

\[ y'' - 4y' + 5y = te^{2t} \]

such that \( y(0) = y'(0) \).
3. Suppose $y_1$ is a solution to a second-order linear differential equation $y'' + p(x)y' + q(x)y = g(x)$, and $y_2$ is a solution to the equation $y'' + p(x)y' + q(x)y = h(x)$ with the same left-hand side, but in general a different right-hand side. Prove that $y_1 + y_2$ is a solution to the equation $y'' + p(x)y' + q(x)y = g(x) + h(x)$. 
4. Find a homogeneous third-order linear differential equation with constant coefficients that has

\[ y(x) = 3e^{-x} - \cos(2x) \]

as a solution. Explain how you found it.

What is the general solution of that differential equation?
6. (a) Let \( f(x) = x^2 + 1 \) be defined on the interval \([0, \pi]\). Find the Fourier sine series of \( f \), and graph the function it converges to on the interval \([-\pi, \pi]\). Be sure to indicate on the graph the value of the sine series at all points in \([-\pi, \pi]\). You do not have to evaluate any integrals. Your answer may have coefficients \( c_n \) in it, along with formula(s) \( c_n = \cdots \int_a^b \cdots dx \).
(b) Find a formal solution to the heat problem given below. Again, you do not need to evaluate any integrals, but you should provide the formula for any fundamental solutions \( u_n \) that you use. Provide a plot of \( u(x, t) \) vs. \( x \) for various values of \( t \), then \( u(x, t) \) vs. \( t \) for various values of \( x \).

\[
\begin{align*}
  u_t &= 4u_{xx} \quad &0 < x < \pi, \quad t > 0, \\
  u(0, t) &= u(\pi, t) = 0 \quad &t > 0, \\
  u(x, 0) &= x^2 + 1 \quad &0 < x < \pi.
\end{align*}
\]
7. If $u$ is a function of the variables $x$ and $t$, consider the PDE $u_{xx} + u_t + u_{tt} = 0$.

(a) If $u(x, t) = X(x)T(t)$ solves the PDE, derive ODEs (sharing a common constant) that $X$ and $T$ would have to satisfy.

(b) Use part (a) to give an example of a non-trivial solution to the PDE.
8. **This is an extra credit question.** Only if a substantial portion of your response to this question is correct will you receive any credit for your work, so it is recommended you try it after having finished the rest of the exam.

The Dirac delta function \( \delta(x) \) can be described as a function that equals zero everywhere except at zero, where it has a singularity. It satisfies the following integral property:

\[
\int_{a}^{b} \delta(x)f(x)\,dx = f(0)
\]

for any continuous function \( f(x) \) and any open interval \((a, b)\) that contains zero.

(a) Find the Fourier series representation of period 1 for the delta function.

(b) One of the following statements about the Fourier series you found in part (a) is correct. Choose the right one and prove it.

- The series converges to \( \delta(x) \) at all points.
- The series diverges at infinitely many points.