of $n$ sides. This polygon is inscribed in the unit circle centered at the origin, and it has one vertex at the point corresponding to the root $z = 1$ $(k = 0)$. If we write

$$\omega_n = \exp \left( \frac{2\pi i}{n} \right),$$

then by property (2), the $n$th roots of unity are simply

$$1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}.$$ Note that $\omega_n^0 = 1$. See Fig. 9 for the interpretation of the three cube roots of unity as the vertices of an equilateral triangle. Figure 10 illustrates the case $n = 6$.

The above method can be used to find the $n$th roots of any nonzero complex number $z_0 = r_0 \exp(i\theta_0)$. Those roots, which are obtained by solving the equation

$$z^n = z_0$$

for $z$, are the numbers

$$c_k = \sqrt[n]{r_0} \exp \left[ i \left( \frac{\theta_0 + 2k\pi}{n} \right) \right] \quad (k = 0, 1, \ldots, n-1),$$

where $\sqrt[n]{r_0}$ denotes the positive $n$th root of $r_0$. The number $\sqrt[n]{r_0}$ is the length of each of the radius vectors representing the $n$ roots. An argument of the first root $c_0$ is $\theta_0/n$, and arguments of the other roots are obtained by adding integral multiples of $2\pi/n$. Consequently, as was the case with the $n$th roots of unity, the roots when $n = 2$ always lie at the opposite ends of a diameter of a circle, one root being the negative of the other; and when $n = 3$, they lie at the vertices of a regular polygon of $n$ sides.

If $c$ is any particular $n$th root of $z_0$, the set of all $n$th roots can be written

$$c, c\omega_n, c\omega_n^2, \ldots, c\omega_n^{n-1},$$

where $\omega_n = \exp(2\pi i/n)$, as defined in equation (6). This is because multiplication of any nonzero complex number by $\omega_n$ corresponds to increasing the argument of that number by $2\pi/n$.

We shall let $z_0^{1/n}$ denote the set of $n$th roots of a nonzero complex number $z_0$. If, in particular, $z_0$ is a positive real number $r_0$, the symbol $r_0^{1/n}$ denotes a set of roots and the symbol $\sqrt[n]{r_0}$ in expression (8) is reserved for the one positive root. When the value of $\theta_0$ that is used in expression (8) is the principal value of $\arg z_0$ ($-\pi < \theta_0 < \pi$), the number $c_0$ is often referred to as the principal $n$th root of $z_0$. Thus when $z_0$ is a positive real number $r_0$, its principal root is $\sqrt[n]{r_0}$.

Note that if $z_0 = 0$, equation (7) has only the solution $z = 0$. Hence the only root at origin is zero.

Finally, a convenient way to remember expression (8) is to write $z_0$ in its most general exponential form (see Sec. 6),

$$z_0 = r_0 \exp[i(\theta_0 + 2k\pi)] \quad (k = 0, \pm 1, \pm 2, \ldots)$$

and to formally apply laws of fractional exponents for real numbers, keeping in mind that there are precisely $n$ distinct roots:

$$z_0^{1/n} = \sqrt[n]{r_0} \exp \frac{i(\theta_0 + 2k\pi)}{n} \quad (k = 0, 1, 2, \ldots, n - 1)$$

This formula is, of course, valid when $n = 1$ as well as when $n = 2, 3, \ldots$. The case $n = 1$ was excluded from our discussion only because of its trivial nature.

**Example 2.** To illustrate formula (10), let us find all values of $(-8i)^{1/3}$, or the three cube roots of $-8i$. We need only write

$$-8i = 8 \exp \left[ i \left( -\frac{\pi}{2} + 2k\pi \right) \right] \quad (k = 0, \pm 1, \pm 2, \ldots),$$

to see that the desired roots are

$$c_k = 2 \exp \left[ i \left( -\frac{\pi}{6} + \frac{2k\pi}{3} \right) \right] \quad (k = 0, 1, 2).$$

In cartesian coordinates, then,

$$c_0 = 2 \exp \left[ i \left( -\frac{\pi}{6} \right) \right] = 2 \left( \cos \frac{\pi}{6} - i \sin \frac{\pi}{6} \right) = \sqrt{3} - i;$$

and, in like manner, we find that $c_1 = 2i$ and $c_2 = -\sqrt{3} - i$. These roots lie at the vertices of an equilateral triangle that is inscribed in a circle of radius 2 centered at the origin. The principal root is $c_0 = \sqrt{3} - i$.

**EXERCISES**

1. Find one value of $\arg z$ when

$$z = \frac{-2}{1 + \sqrt{3}i}; \quad \frac{i}{2 - 2i}; \quad (\sqrt{3} - i)^6.$$

*Ans.* (a) $2\pi/3$; (c) $\pi$.

2. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to cartesian coordinates, show that

$$i(1 - \sqrt{3}i)(\sqrt{3} + i) = 2(1 + \sqrt{3}i); \quad 5i/(2 + i) = 1 + 2i;$$

$$(-1 + i)^7 = -8(1 + i); \quad (1 + \sqrt{3}i)^{-10} = 2^{-11}(-1 + \sqrt{3}i).$$