5.3.21) We can compute

\[
\frac{1}{4} ||\mathbf{u} + \mathbf{v}||^2 - \frac{1}{4} ||\mathbf{u} - \mathbf{v}||^2
\]

\[
= \frac{1}{4} (\langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle - \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle)
\]

\[
= \frac{1}{4} (\langle \mathbf{u}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle - \langle \mathbf{u}, \mathbf{u} - \mathbf{v} \rangle - \langle \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle)
\]

\[
= \frac{1}{4} (\langle \mathbf{u} + \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{u} + \mathbf{v}, \mathbf{v} \rangle - \langle \mathbf{u} - \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{u} - \mathbf{v}, \mathbf{v} \rangle)
\]

\[
= \frac{1}{4} (\langle \mathbf{u} + \mathbf{v} - (\mathbf{u} - \mathbf{v}), \mathbf{u} \rangle + \langle \mathbf{u} + \mathbf{v} - (\mathbf{u} - \mathbf{v}), \mathbf{v} \rangle)
\]

\[
= \frac{1}{4} (2\langle \mathbf{v}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle)
\]

\[
= \frac{1}{4} (2\langle \mathbf{u}, \mathbf{v} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle)
\]

\[
= \frac{1}{4} (4\langle \mathbf{u}, \mathbf{v} \rangle)
\]

\[
= (\langle \mathbf{u}, \mathbf{v} \rangle).
\]

5.4.33) Call the set in question \( S \). It is immediate from the definition that \( S \) is a subset of \( \mathbb{R}^n \). By Theorem 4.3, to show that it is a subspace, it suffices to show that it is closed under addition and scalar multiplication.

For addition, let \( \mathbf{v}, \mathbf{w} \in S \). This means that \( \langle \mathbf{v}, \mathbf{u} \rangle = 0 \) and \( \langle \mathbf{w}, \mathbf{u} \rangle = 0 \). Thus,

\[
\langle \mathbf{v} + \mathbf{w}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle = 0 + 0 = 0,
\]

so \( \mathbf{v} + \mathbf{w} \in S \).

For scalar multiplication, let \( \mathbf{v} \in S \) and \( c \in \mathbb{R} \). Since \( \mathbf{v} \in S \), we have \( \langle \mathbf{v}, \mathbf{u} \rangle = 0 \). This lets us compute

\[
\langle c\mathbf{v}, \mathbf{u} \rangle = c\langle \mathbf{v}, \mathbf{u} \rangle = c0 = 0,
\]

from which \( c\mathbf{v} \in S \).